Relic Density at NLO
the Thermal IR Finiteness

Andrzej Hryczuk

University of Oslo*

based on: M. Beneke, F. Dighera, AH, 1409.3049
M. Beneke, F. Dighera, AH, 1607.03910

Rethinking QTF, DESY, 28th September 2016

*on leave from National Centre for Nuclear Research, Warsaw, Poland

Dark Matter at NLO

Bergstrom '89; Drees et al., 9306325;
Ullio & Bergstrom, 9707333 — helicity suppression lifting

⋮

Bergstrom et al., 0507229;
Bringmann et al., 0710.3169 — spectral features in indirect searches

⋮

Ciafaloni et al., 1009.0224
Cirelli et al., 1012.4515
Ciafaloni et al., 1202.0692
AH & Iengo, 1111.2916 — large EW corrections

⋮

Chatterjee et al., 1209.2328
Harz et al., 1212.5241
Ciafaloni et al., 1305.6391
Hermann et al., 1404.2931
Boudjema et al., 1403.7459 — ***thermal relic density***

⋮

$$\Omega_{DM} h^2 = 0.1187 \pm 0.0017.$$

<1.5% uncertainty!

Planck+WMAP pol.+highL+BAO; 1303.5062

⋮

SloopS, DM@NLO, PPC4DMID — NLO codes

Relic Density

standard approach

T

$\Gamma_{\text{ann}} > H$ DM in equilibrium

$\Gamma_{\text{ann}} \sim H$ chemical decoupling time

$\Gamma_{\text{ann}} < H$ freeze-out

non-equilibrium

time evolution of $f_\chi(p)$ in kinetic theory:

$$\boxed{E\left(\partial_t - H\vec{p}\cdot\nabla_{\vec{p}}\right)} f_\chi = \mathcal{C}[f_\chi] \quad \Rightarrow \quad \frac{dn_\chi}{dt} + 3Hn_\chi = C$$

Liouville operator in FRW background

the collision term integrated

assumptions for using Boltzmann eq: classical limit, molecular chaos,...

Relic Density

Boltzmann eq.

$$\frac{dn_\chi}{dt} + 3Hn_\chi = -\langle \sigma_{\chi\bar{\chi}\to ij}\sigma_{\rm rel}\rangle^{\rm eq}\left(n_\chi n_{\bar{\chi}} - n_\chi^{\rm eq} n_{\bar{\chi}}^{\rm eq}\right)$$

Re-written for the comoving number density:

$$\frac{dY}{dx} = \sqrt{\frac{g_* \pi m_\chi^2}{45G}}\,\frac{\langle \sigma_{\chi\bar{\chi}\to ij}\sigma_{\rm rel}\rangle^{\rm eq}}{x^2}\left(Y^2 - Y_{\rm eq}^2\right)$$

$$\lim_{x\to 0} Y = Y_{\rm eq} \qquad \lim_{x\to\infty} Y = \text{const}$$

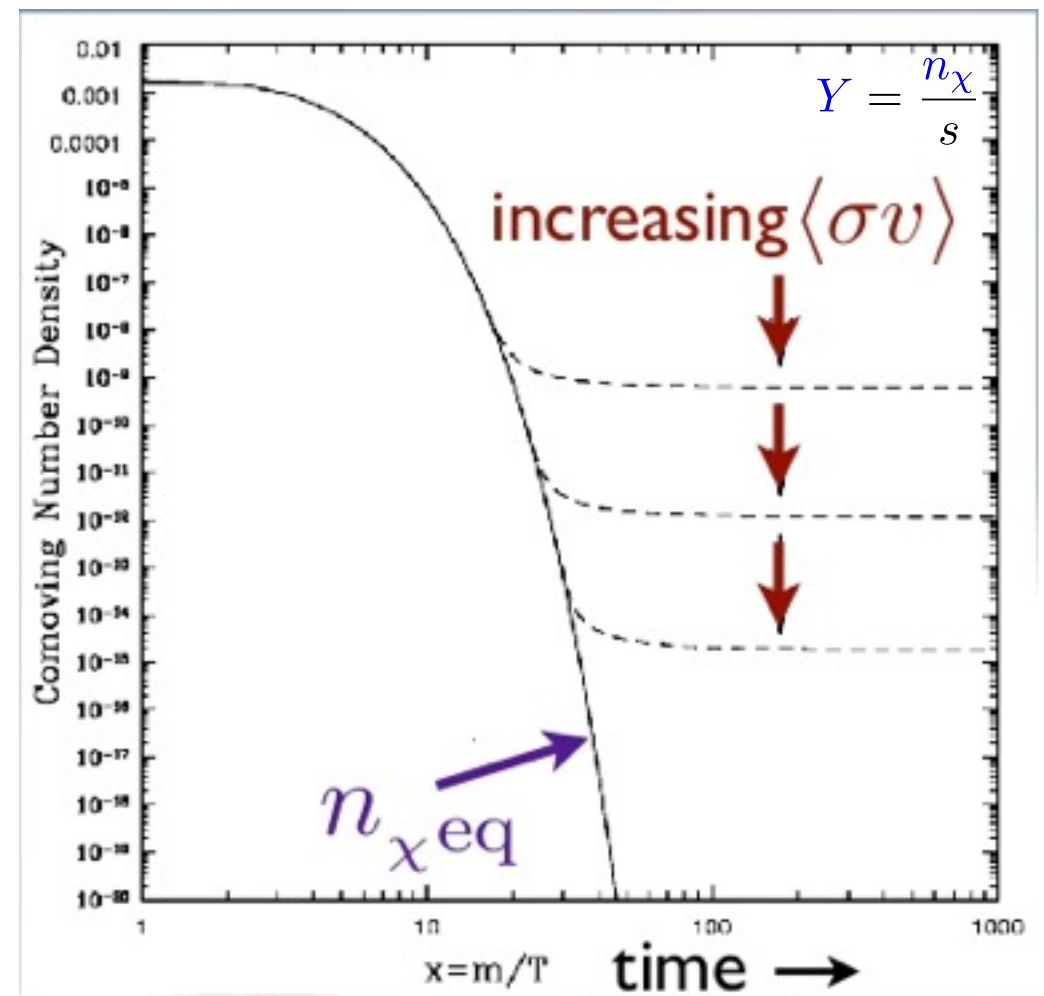

Fig.: Jungman, Kamionkowski & Griest, PR'96

Recipe:
compute LO annihilation cross-section,
take a thermal bath average,
plug in to BE… and voilà

Relic Density at NLO

Recall at LO:

$$C_{\rm LO} = -h_\chi^2 \int \frac{d^3\vec{p}_\chi}{(2\pi)^3}\frac{d^3\vec{p}_{\bar\chi}}{(2\pi)^3}\ \sigma_{\chi\bar\chi\to ij}v_{\rm rel}\ \left[f_\chi f_{\bar\chi}(1\pm f_i)(1\pm f_j) - f_i f_j(1\pm f_\chi)(1\pm f_{\bar\chi})\right]$$

crucial point: $\boxed{p_\chi + p_{\bar\chi} = p_i + p_j \Rightarrow f_\chi^{\rm eq} f_{\bar\chi}^{\rm eq} \approx f_i^{\rm eq} f_j^{\rm eq}}$

in Maxwell approx.

at NLO both virtual one-loop and 3-body processes contribute:

$$C_{1-{\rm loop}} = -h_\chi^2 \int \frac{d^3\vec{p}_\chi}{(2\pi)^3}\frac{d^3\vec{p}_{\bar\chi}}{(2\pi)^3}\ \sigma^{1-{\rm loop}}_{\chi\bar\chi\to ij}v_{\rm rel}\ \left[f_\chi f_{\bar\chi}(1\pm f_i)(1\pm f_j) - f_i f_j(1\pm f_\chi)(1\pm f_{\bar\chi})\right]$$

$$C_{\rm real} = -h_\chi^2 \int \frac{d^3\vec{p}_\chi}{(2\pi)^3}\frac{d^3\vec{p}_{\bar\chi}}{(2\pi)^3}\ \sigma_{\chi\bar\chi\to ij\gamma}v_{\rm rel}\ \left[f_\chi f_{\bar\chi}(1\pm f_i)(1\pm f_j)(1+f_\gamma) - f_i f_j f_\gamma(1\pm f_\chi)(1\pm f_{\bar\chi})\right]$$

RELIC DENSITY AT NLO

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$$p_\chi + p_{\bar\chi} = p_i + p_j \pm p_\gamma \Rightarrow$$

photon can be arbitrarily soft
$f_\gamma \sim \omega^{-1}$

Maxwell approx. not valid anymore...

…even bigger problem: *T*-dependent IR divergence!

Relic Density

what Really happens at NLO?

$$
\begin{aligned}
C_{\rm NLO} \sim \int d\Pi_{\chi\bar{\chi}ij} \;\; f_\chi f_{\bar\chi} \;\; \Big\{ & |\mathcal{M}^{\rm LO}_{\chi\bar\chi\to ij}|^2 + |\mathcal{M}^{{\rm NLO}\;T=0}_{\chi\bar\chi\to ij}|^2 + \int d\Pi_\gamma |\mathcal{M}_{\chi\bar\chi\to ij\gamma}|^2 + \\
& |\mathcal{M}^{{\rm NLO}\;T\neq0}_{\chi\bar\chi\to ij}|^2 + \int d\Pi_\gamma \left[f_\gamma \left(|\mathcal{M}_{\chi\bar\chi\to ij\gamma}|^2 + |\mathcal{M}_{\chi\bar\chi\gamma\to ij}|^2 \right) \right. \\
& \left. - f_i \left(|\mathcal{M}_{\chi\bar\chi\to ij\gamma}|^2 + |\mathcal{M}_{\chi\bar\chi i\to j\gamma}|^2 \right) - f_j \left(|\mathcal{M}_{\chi\bar\chi\to ij\gamma}|^2 + |\mathcal{M}_{\chi\bar\chi j\to i\gamma}|^2 \right) \right] \Big\} \\
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\end{aligned}
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photon absorption

thermal 1-loop

photon emission

SM fermions emission

SM fermions absorption

$$
\begin{aligned}
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\end{aligned}
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QUESTIONS:

Beneke, Dighera, AH, 1409.3049

1. how the (soft and collinear) IR divergence cancellation happen?

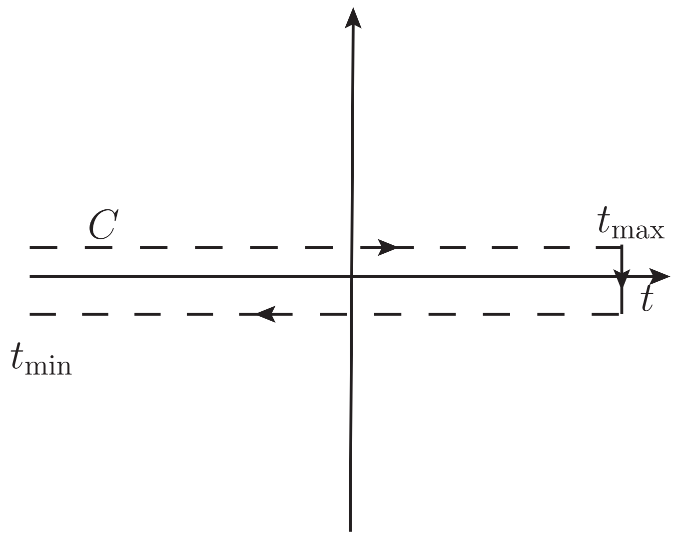
2. does Boltzmann equation itself receive quantum corrections?

3. how large are the remaining finite T corrections?

Program: develop a method for relic density calculation directly from QFT and free from IR problems

framework exists: non-equilibrium thermal field theory

CLOSED TIME PATH
FORMALISM



$$i\Delta(x,y) = \langle T_C \phi(x)\phi^\dagger(y)\rangle,$$
$$iS_{\alpha\beta}(x,y) = \langle T_C \psi_\alpha(x)\bar{\psi}_\beta(y)\rangle,$$

contour Green's functions obey Dyson-Schwinger eqs:

$$\Delta(x,y) = \Delta_0(x,y) - \int_C d^4z \int_C d^4z' \Delta_0(x,z)\Pi(z,z')\Delta(z',y),$$
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which can be rewritten in the form of Kadanoff-Baym eqs:

$$(-\partial^2 - m_\phi^2)\Delta^{\lessgtr}(x,y) - \int d^4z \left(\Pi_h(x,z)\Delta^{\lessgtr}(z,y) - \Pi^{\lessgtr}(x,z)\Delta_h(z,y)\right) = \mathcal{C}_\phi,$$
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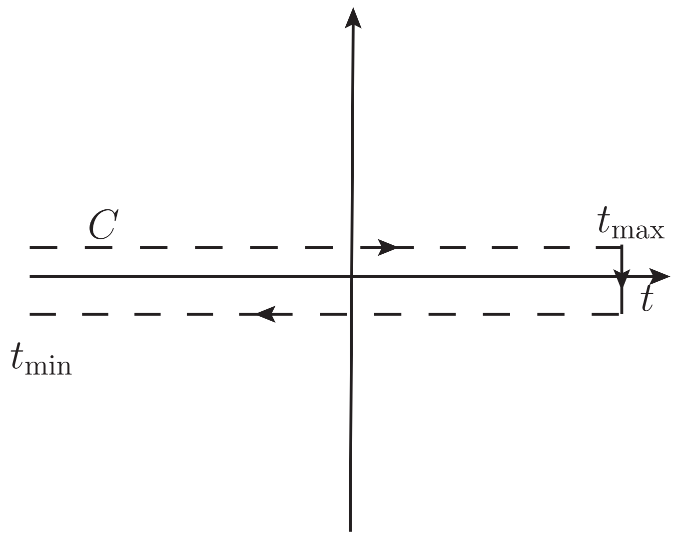
Closed Time Path
formalism



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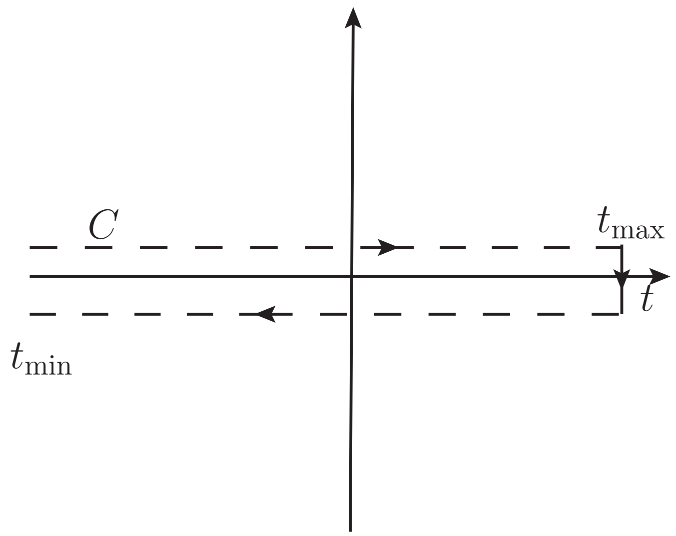

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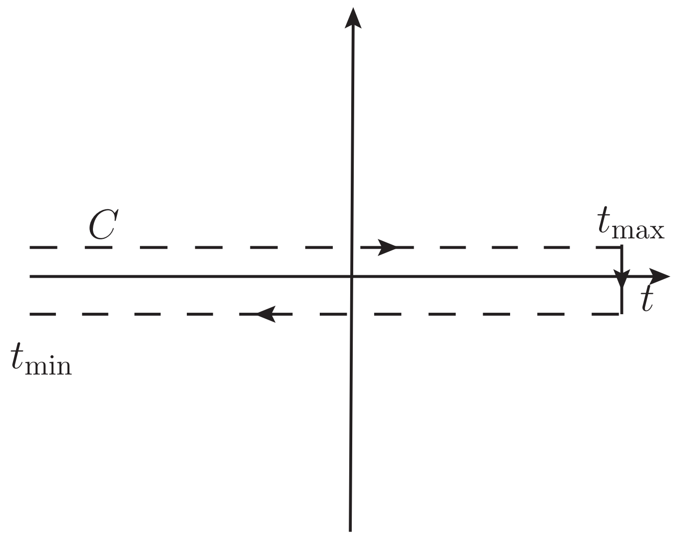
Closed Time Path

FORMALISM



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CLOSED TIME PATH

FORMALISM: COLLISION TERM

the fermion collision terms is defined as:

propagators

$$\mathcal{C}_\chi = \frac{1}{2}\int d^4z\, \left(\Sigma^>(x,z)S^<(z,y) - \Sigma^<(x,z)S^>(z,y)\right)$$

self-energies

where the propagators:

thermal part

$$iS^c(p) = \frac{i\left(\not{p}+m\right)}{p^2-m^2+i\eta} - \boxed{2\pi\left(\not{p}+m\right)\delta\left(p^2-m^2\right)f\left(p^0\right)}$$

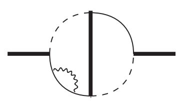
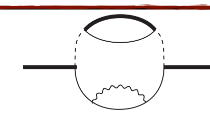
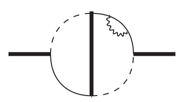
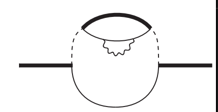
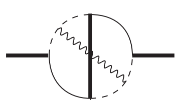
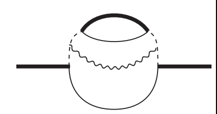
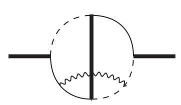
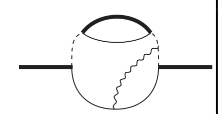
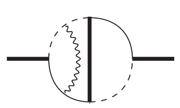
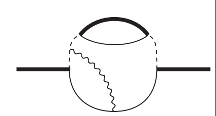
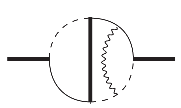
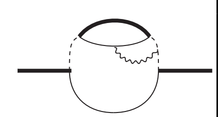
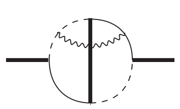
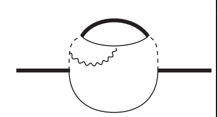
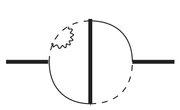
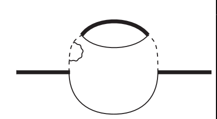
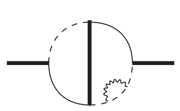
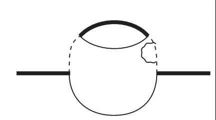
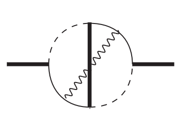
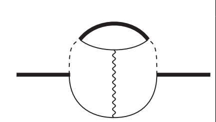
$$iS^a(p) = -\frac{i\left(\not{p}+m\right)}{p^2-m^2+i\eta} + 2\pi\left(\not{p}+m\right)\delta\left(p^2-m^2\right)\left(1-f\left(p^0\right)\right)$$

$$\left.\begin{aligned} iS^>(p) &= 2\pi\left(\not{p}+m\right)\delta\left(p^2-m^2\right)\left(1-f\left(p^0\right)\right) \\ iS^<(p) &= -2\pi\left(\not{p}+m\right)\delta\left(p^2-m^2\right)f\left(p^0\right)\end{aligned}\right\}\ \text{"cut" propagators}$$

the presence of distribution functions inside propagators $\Rightarrow$ known collision term structure

Results

IR divergence cancellation: S-wave

The divergent part J_{-1}							
Type A	Real	Virtual	External	Type B	Real	Virtual	External
	$\frac{\alpha(1-2\epsilon^2)}{\pi\epsilon^2}$		$-\frac{\alpha(1-2\epsilon^2)}{\pi\epsilon^2}$		$-\frac{\alpha}{\pi\epsilon^2}$		$\frac{\alpha}{\pi\epsilon^2}$
	$\frac{\alpha(1-2\epsilon^2)}{\pi\epsilon^2}$		$-\frac{\alpha(1-2\epsilon^2)}{\pi\epsilon^2}$		$-\frac{\alpha}{\pi\epsilon^2}$		$\frac{\alpha}{\pi\epsilon^2}$
	0				0		
	0	0			0	0	
	0	0			0	0	
	0	0			0	0	
	0	0			0	0	
		0				0	
		0				0	
	$\frac{2\alpha(1-2\epsilon^2)^2}{\pi\epsilon^2\sqrt{1-4\epsilon^2}}L$	$-\frac{2\alpha(1-2\epsilon^2)^2}{\pi\epsilon^2\sqrt{1-4\epsilon^2}}L$			$\frac{2\alpha(1-2\epsilon^2)}{\pi\epsilon^2\sqrt{1-4\epsilon^2}}L$	$-\frac{2\alpha(1-2\epsilon^2)}{\pi\epsilon^2\sqrt{1-4\epsilon^2}}L$	

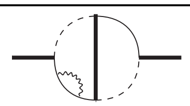
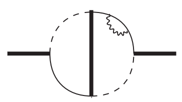
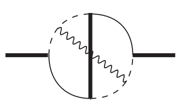
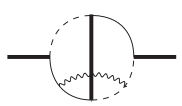
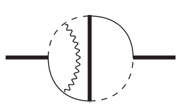
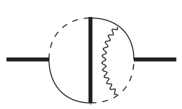
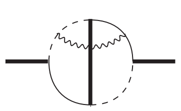
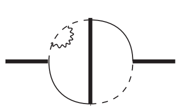
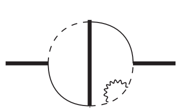
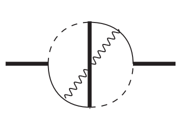
cancels in **every** row separately

$\Longrightarrow$ every CTP self-energy is IR finite

Results

finite T correction: s-wave

factorized $\frac{\pi}{6}\alpha\tau^2\frac{a_{\rm tree}}{\epsilon^2}$

The finite part J_1			
Type A	Real	Virtual	External
	$\frac{2(1-\xi^2)}{D^2D_\xi^2}+\frac{(1-2\epsilon^2)p_1(\epsilon,\xi)}{2D^2D_\xi^2}\boxed{+\frac{1}{2\sqrt{D}}L}$		$\frac{(1-2\epsilon^2)(\xi^2-3D)}{2DD_\xi}\boxed{-\frac{1}{2\sqrt{D}}L}$
	— ” —		— ” —
	$-\frac{4(1-2\epsilon^2)D}{D_\xi^2}$		
	$-\frac{2(1-2\epsilon^2)\xi^2}{D_\xi^2}-\frac{f_1(\epsilon,\xi)}{\sqrt{D}D_\xi^2}L$	$\frac{2(1-2\epsilon^2)(D-\xi^2)}{D_\xi^2}+\frac{f_1(\epsilon,\xi)}{\sqrt{D}D_\xi^2}L$	
	— ” —	— ” —	
	— ” —	— ” —	
	— ” —	— ” —	
		$-\frac{4(1-2\epsilon^2)D}{D_\xi^2}$	
		— ” —	
	$\frac{2(1-2\epsilon^2)p_2(\epsilon,\xi)+(1-\xi^2)^2}{D^2D_\xi^2}+\frac{4f_2(\epsilon,\xi)}{\sqrt{D}D_\xi^2}L$	$\frac{16\epsilon^2(2-3\epsilon^2)-(3-\xi^2)^2}{D_\xi^2}-\frac{4f_2(\epsilon,\xi)}{\sqrt{D}D_\xi^2}L$	

Log terms cancels in **every** row separately

$\Downarrow$

no collinear divergence!

$$\xi=\frac{m_\phi}{m_\chi}\gtrsim 1$$

$$\tau=\frac{T}{m_\chi}\ll 1$$

$$\epsilon=\frac{m_f}{2m_\chi}\ll\tau$$

separate contributions complicated, but when summed up...

$$\Delta a_{\tau^4}^{\epsilon=0}=\frac{8\pi^2\lambda^4\alpha\tau^4}{45}\frac{1}{(1+\xi^2)^4}=\frac{4\pi}{45}\alpha\tau^4\frac{1}{(1+\xi^2)^2}\frac{a_{\rm tree}}{\epsilon^2}\Big|_{\epsilon=0}$$

strongly suppressed as at kinetic equilibrium $\tau\sim v^2$

The power of Thermal OPE

M. Beneke, F. Dighera, AH, 1607.03910

The cross section can be written as the Im part of the forward scattering amplitude:

$$\sigma v_{\rm rel} = \frac{2}{s}\,\mathrm{Im}\left\{(-i)\boxed{\int d^4x\,\frac{1}{4}\sum_{\rm spin}\langle\bar\chi\chi;T|\,\mathcal{T}\left\{\mathcal{O}_{\rm ann}(0)\,\mathcal{O}^\dagger_{\rm ann}(x)\right\}\,|\bar\chi\chi;T\rangle}\right\}$$

clear separation of soft (thermal effects) and hard (annihilation/decay) modes

$$T \ll m$$

$\Rightarrow$ **Operator Product Expansion**

$$-i\int d^4x\; e^{-ip\cdot x}\,\mathcal{T}\left\{J^\mu_A(0)\,J^{\nu\dagger}_B(x)\right\} = \sum_i C^i_{AB}(p)\cdot\mathcal{O}_i$$

Wilson coeffs. matched at T=0

Possible operators up to dim 4:

$$\mathbb{1}\;,\qquad F^{\alpha\beta}F^{\gamma\delta}\;,\qquad m_f\bar f\,\Gamma f\;,\qquad \bar f\,\Gamma\, iD^\alpha f$$

Matrix elements: LO $\qquad \mathcal{O}(\alpha T^4) \qquad \mathcal{O}(\alpha m_f^2 T^2) \qquad \mathcal{O}(\alpha T^4)$

No dim 2 operator!

No IR divergence to begin with!

ADVANTAGES OF OPE

- The scaling with T is manifest
- Separation of T=0 and T-dependent contributions
- Significant simplification of the computations
- Clear physics interpretation: at $\mathcal{O}(\alpha\tau^2)$ effects of thermal kinetic energy

Example: muon decay in thermal bath*

Czarnecki et al. '11

$$-i\int d^4x\; e^{-ip\cdot x}\;\mathcal{T}\{J^\mu(0)\,J^{\nu\dagger}(x)\} = C_0^{\mu\nu}\bar\psi\psi + C_2^{\mu\nu}\cancel{\bar\psi\frac{i}{2}\sigma_{\alpha\beta}}F^{\alpha\beta}\psi + \mathcal{O}(m_\psi^{-3}),$$

*Analogy: semi-leptonic H_b decay in QCD

In the Literature:

OPE in finite temperature - Hatsuda, Koike, Lee '93; Mallik '97; …

Related EFT approach - Biondini, Brambilla, Escobedo, Vairo '13; …

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$$\bar{\psi}\psi = \bar{\psi}\not{v}\psi + \boxed{\frac{1}{2m_\psi^2}\bar{\psi}\,(iD_\perp)^2\,\psi} + \cancel{\frac{i}{4m_\psi^2}\bar{\psi}\sigma_{\alpha\beta}F^{\alpha\beta}\psi} + \mathcal{O}(m_\psi^{-3}),$$

LO $\qquad \mathcal{O}(\alpha\tau^2)$

*Analogy: semi-leptonic H_b decay in QCD

…and the final correction:

$$\Gamma_T = \Gamma_0\,(1 - \boxed{K_\psi}) + \mathcal{O}(T^3/m_\psi^3).$$

In the Literature:

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Conclusions

1. how the (soft and collinear) IR divergence cancellation happen?
   automatic in thermal QFT formalism, cancellation at the level of every CTP self-energy

2. does Boltzmann equation itself receive quantum corrections?
   no, not at NLO

3. how large are the remaining finite T corrections?
   strongly suppressed, of order $\mathcal{O}(\alpha T^4)$

4. the thermal OPE method provides a useful tool and also physics interpretation of the thermal correction

To take home:

complete framework for computations of relic density at NLO w/ thermal effects

BACKUP SLIDES

CLOSED TIME PATH

PATH TO BOLTZMANN EQUATION

Kadanoff-Baym $\Longrightarrow$ Boltzmann

$$E\left(\partial_t - H\vec{p}\cdot\nabla_{\vec{p}}\right) f = \mathcal{C}[f].$$

collision term **derived** from thermal QFT

Assumptions:

weak inhomogeneity

gradient expansion

quasi-particle approx.

weak interactions

Justification:

inhomogeneity

plasma excitation momenta

$$\partial \ll k$$

freeze-out happens close to equilibrium

RESULTS

coming back to our example...

every contribution can be written in a form:

$$\int_0^\infty d\omega\, f_\gamma(\omega) \boxed{S(\omega, e_\chi, \epsilon, \xi)} \qquad f_\gamma(\omega) = \frac{1}{1 - e^{\omega/T}}$$

photon energy

expand in ω

$$S = \sum_{i=-1}^{\infty} s_n \omega^n$$

note:

$$J_n \equiv \int_0^\infty f_B(\omega)\omega^n d\omega = \begin{cases} \text{div} & n \leq 0 \\ \sim \tau^{n+1} & n > 0 \end{cases}$$

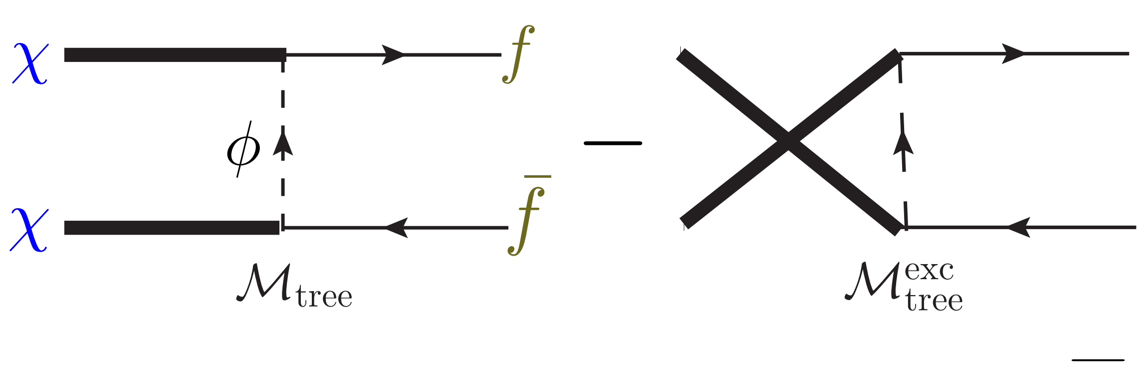
IR divergence in separate terms: $\begin{array}{rcl} J_{-1} & \leftrightarrow & T = 0 \text{ soft div} \\ J_0 & \leftrightarrow & T = 0 \text{ soft eikonal} \end{array}$

finite T corrections: $J_1 \leftrightarrow \mathcal{O}(\tau^2) \quad \ldots$

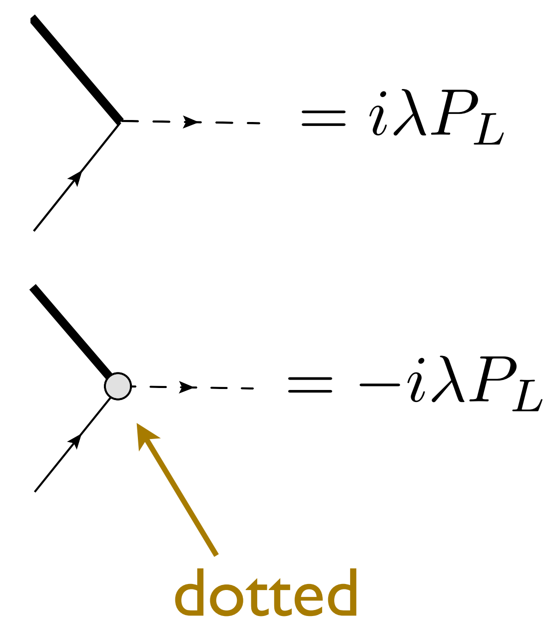
Collision Term
example

Bino-like DM: χ Majorana fermion, SM singlet

annihilation process at tree level:



vertices (2 types):



scale hierarchy: $\boxed{m_\phi \gtrsim m_\chi} \gg T \gg \boxed{m_f}$

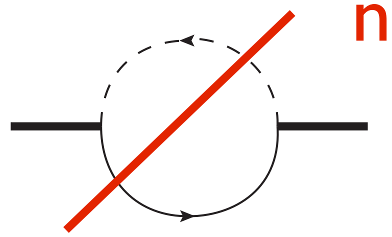
no thermal contributions

effectively massless

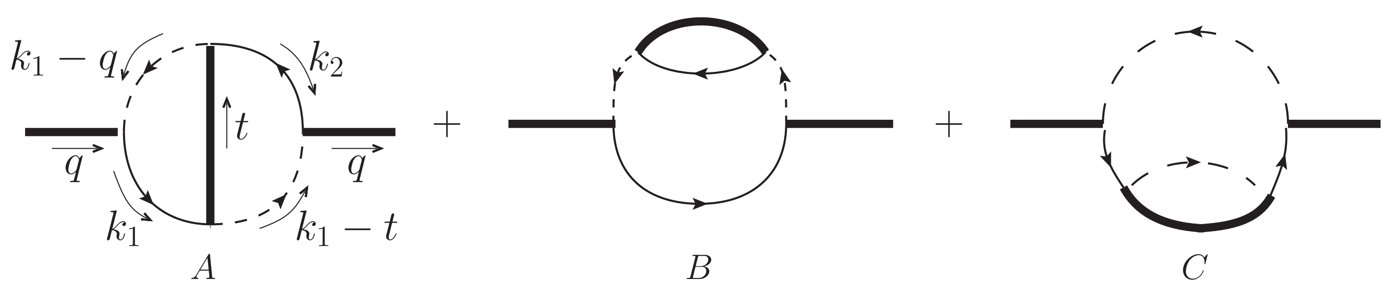
rescaled variables: $\tau = \dfrac{T}{m_\chi} \ll 1 \qquad \epsilon = \dfrac{m_f}{2m_\chi} \ll \tau \qquad \xi = \dfrac{m_\phi}{m_\chi} \gtrsim 1$

COLLISION TERM
COMPUTATION

no # changing processes

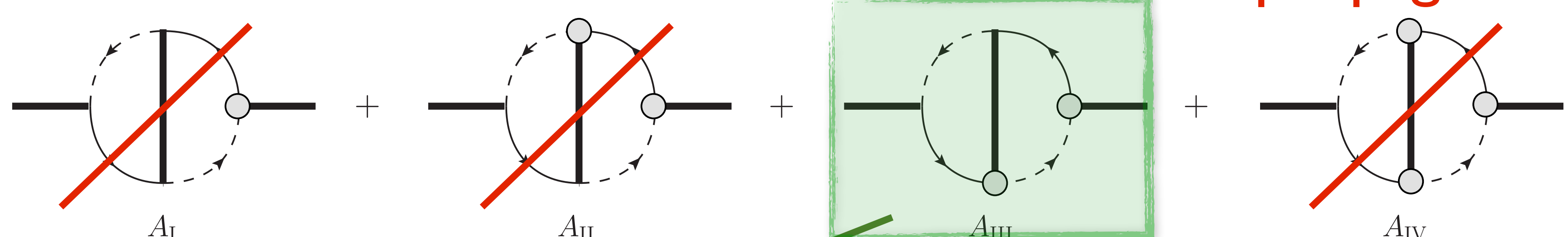
$i\Sigma_1 =$ 

summed over dotted and undotted indices

$i\Sigma_2 =$ 

$\Downarrow$

cut scalar propagator

$i\Sigma^{>}_{A} =$ 

$$\Sigma^{>}_{A_{\mathrm{III}}}(q)\, S^{<}(q) = -\lambda^4 \int \frac{d^4 t}{(2\pi)^4} \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} (2\pi)^4 \delta(q + t - k_1 - k_2)$$
$$\underbrace{i\Delta^{11}(k_1 - q)\, i\Delta^{22}(k_1 - t)}_{\equiv \mathcal{S}} \underbrace{P_{\mathrm{R}} iS^{21}(k_2)\, P_{\mathrm{L}} iS^{12}(t)\, P_{\mathrm{L}} iS^{21}(k_1)\, P_{\mathrm{R}} iS^{12}(q)}_{\equiv \mathcal{F}}$$

COLLISION TERM
MATCHING

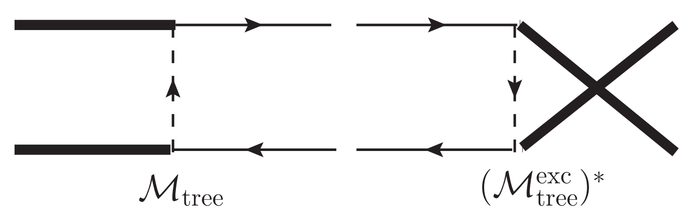
after inserting the propagators:

$$\Sigma^{>}_{A_{\rm III}}(q)\, S^{<}(q) = \frac{1}{2E_{\chi_1}}(2\pi)\,\delta\left(q^0 - E_{\chi_1}\right)\int \frac{d^4 t}{(2\pi)^3\, 2E_{\chi_2}}\delta\left(t^0 - E_{\chi_2}\right)\times$$

$$\int \frac{d^3\vec{k}_1}{(2\pi)^3\, 2E_{f_1}}\frac{d^3\vec{k}_2}{(2\pi)^3\, 2E_{f_2}}(2\pi)^4\,\delta\left(q + t - k_1 - k_2\right)|\mathcal{M}_A|^2\left[f_\chi(q)\, f_\chi(t)\left(1 - f_f^{\rm eq}\left(k_1^0\right)\right)\left(1 - f_f^{\rm eq}\left(k_2^0\right)\right)\right]$$

$\Longrightarrow$ one indeed recovers the known collision term and

$|\mathcal{M}_A|^2 =$



(part of) tree level $|\mathcal{M}|^2$

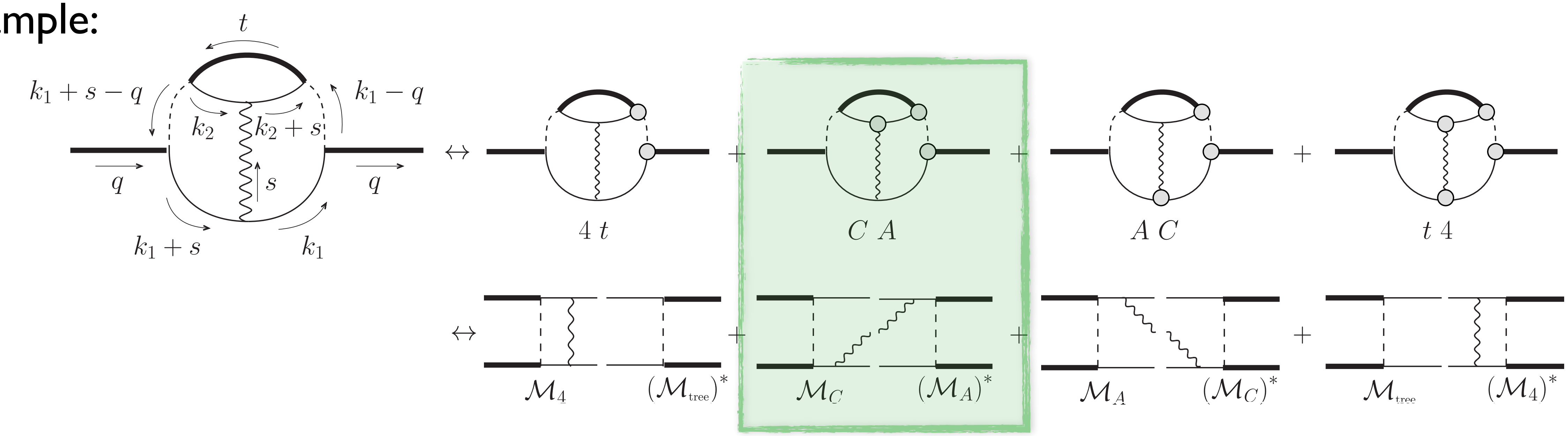
repeating the same for B type diagrams the bottom line:

$i\Sigma^{>} \leftrightarrow$ tree level annihilation contribution to the collision term

COLLISION TERM
MATCHING AT NLO

$i\Sigma_3 =$ 20 self-energy diagrams

example:



$$\Sigma^{>}_{\rm CA}(q)\,S^{<}(q) = \frac{1}{2E_{\chi_1}}(2\pi)\,\delta\left(q^0 - E_{\chi_1}\right)\int \frac{d^4t}{(2\pi)^3\,2E_{\chi_2}}\delta\left(t^0 - E_{\chi_2}\right)$$

$$\int \frac{d^3\vec{k}_1}{(2\pi)^3\,2E_{f_1}}\frac{d^3\vec{k}_2}{(2\pi)^3\,2E_{f_2}}\frac{d^3\vec{s}}{(2\pi)^3\,2E_{\gamma}}(2\pi)^4\,\delta\left(q+t-k_1-k_2-s\right)$$

$$\mathcal{M}_C\left(\mathcal{M}_A\right)^*\left[f_\chi(q)\,f_\chi(t)\left(1-f_f^{\rm eq}\left(k_1^0\right)\right)\left(1-f_f^{\rm eq}\left(k_2^0\right)\right)\left(1+f_\gamma^{\rm eq}\left(s^0\right)\right)\right]$$

$\Rightarrow$ at NLO thermal effects do **not** change the collision therm structure