Relic Density at NLO

the Thermal Corrections

Andrzej Hryczuk

University of Oslo*

based on: M. Beneke, F. Dighera, AH, 1409.3049
M. Beneke, F. Dighera, AH, 1607.03910

Dark Side of the Universe, Bergen, 29th July 2016

* on leave from National Centre for Nuclear Research, Warsaw, Poland

Dark Matter at NLO

Bergstrom '89; Drees et al., 9306325;
Ullio & Bergstrom, 9707333 } helicity suppression lifting

⋮

Bergstrom et al., 0507229;
Bringmann et al., 0710.3169 } spectral features in indirect searches

⋮

Ciafaloni et al., 1009.0224
Cirelli et al., 1012.4515
Ciafaloni et al., 1202.0692
AH & Iengo, 1111.2916 } large EW corrections

⋮

Chatterjee et al., 1209.2328
Harz et al., 1212.5241
Ciafaloni et al., 1305.6391
Hermann et al., 1404.2931
Boudjema et al., 1403.7459 } ***thermal relic density***

⋮

$$\Omega_{DM} h^2 = 0.1187 \pm 0.0017.$$

<1.5% uncertainty!

Planck+WMAP pol.+highL+BAO; 1303.5062

⋮

SloopS, DM@NLO, PPC4DMID } NLO codes

Relic Density

Standard Approach

T ↑

$\Gamma_{\rm ann} > H$ — DM in equilibrium

$\Gamma_{\rm ann} \sim H$ — chemical decoupling

$\Gamma_{\rm ann} < H$ — freeze-out

↓ time

non-equilibrium

time evolution of $f_\chi(p)$ in kinetic theory:

$$E\left(\partial_t - H\vec{p}\cdot\nabla_{\vec{p}}\right) f_\chi = \mathcal{C}[f_\chi] \quad \Rightarrow \quad \frac{dn_\chi}{dt} + 3Hn_\chi = C$$

Liouville operator in FRW background

the collision term

integrated

assumptions for using Boltzmann eq: classical limit, molecular chaos,...

Relic Density
Boltzmann eq.

$$\frac{dn_\chi}{dt} + 3Hn_\chi = -\langle \sigma_{\chi\bar{\chi}\to ij}\sigma_{\rm rel}\rangle^{\rm eq}\left(n_\chi n_{\bar{\chi}} - n_\chi^{\rm eq} n_{\bar{\chi}}^{\rm eq}\right)$$

Re-written for the comoving number density:

$$\frac{dY}{dx} = \sqrt{\frac{g_*\pi m_\chi^2}{45G}}\,\frac{\langle \sigma_{\chi\bar{\chi}\to ij}\sigma_{\rm rel}\rangle^{\rm eq}}{x^2}\left(Y^2 - Y_{\rm eq}^2\right)$$

$$\lim_{x\to 0} Y = Y_{\rm eq} \qquad \lim_{x\to\infty} Y = \text{const}$$

Recipe:
compute LO annihilation cross-section,
take a thermal bath average,
plug in to BE… and voilà

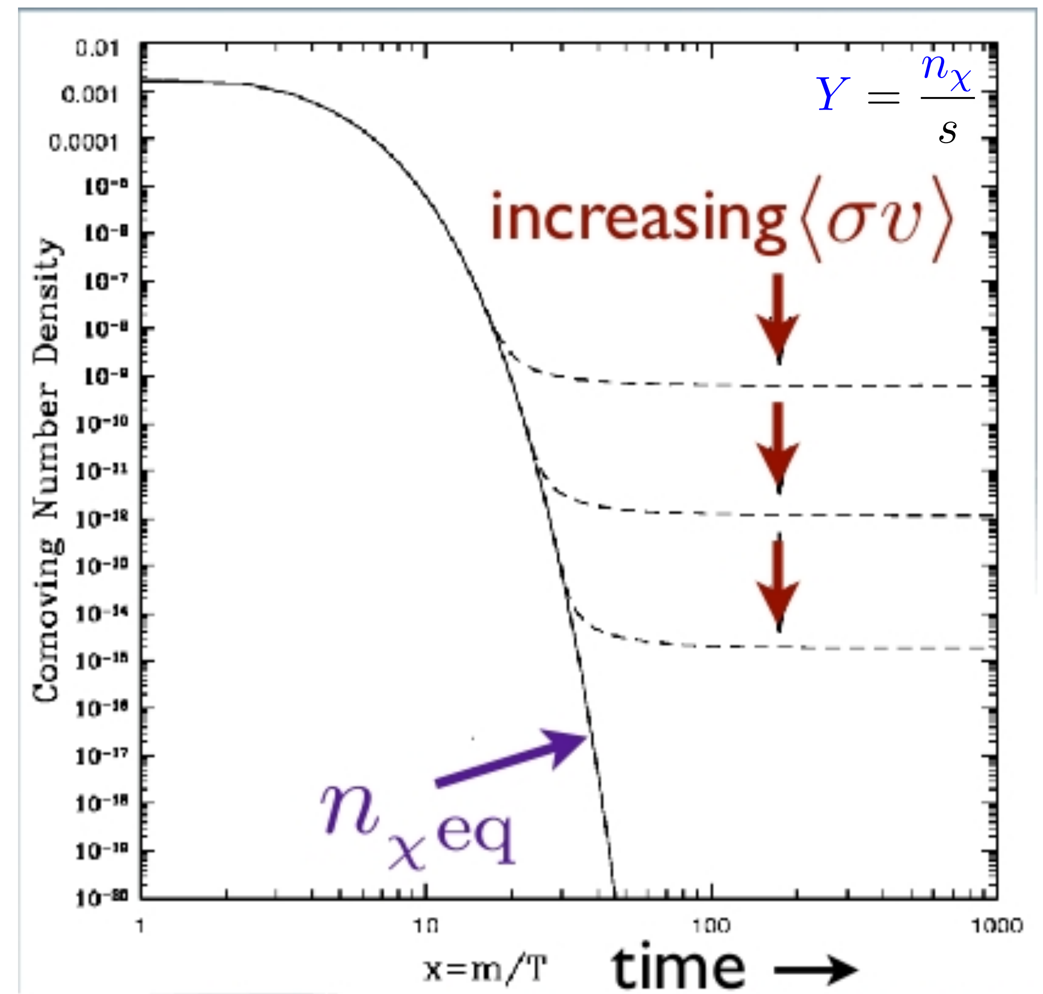

Fig.: Jungman, Kamionkowski & Griest, PR'96

Relic Density at NLO

Recall at LO:

$$C_{\rm LO} = -h_\chi^2 \int \frac{d^3\vec{p}_\chi}{(2\pi)^3}\frac{d^3\vec{p}_{\bar\chi}}{(2\pi)^3}\ \sigma_{\chi\bar\chi\to ij}v_{\rm rel}\ \left[f_\chi f_{\bar\chi}(1\pm f_i)(1\pm f_j) - f_i f_j (1\pm f_\chi)(1\pm f_{\bar\chi})\right]$$

crucial point: $p_\chi + p_{\bar\chi} = p_i + p_j \Rightarrow f_\chi^{\rm eq} f_{\bar\chi}^{\rm eq} \approx f_i^{\rm eq} f_j^{\rm eq}$

in Maxwell approx.

at NLO both virtual one-loop and 3-body processes contribute:

$$C_{1-{\rm loop}} = -h_\chi^2 \int \frac{d^3\vec{p}_\chi}{(2\pi)^3}\frac{d^3\vec{p}_{\bar\chi}}{(2\pi)^3}\ \sigma^{1-{\rm loop}}_{\chi\bar\chi\to ij}v_{\rm rel}\ \left[f_\chi f_{\bar\chi}(1\pm f_i)(1\pm f_j) - f_i f_j (1\pm f_\chi)(1\pm f_{\bar\chi})\right]$$

$$C_{\rm real} = -h_\chi^2 \int \frac{d^3\vec{p}_\chi}{(2\pi)^3}\frac{d^3\vec{p}_{\bar\chi}}{(2\pi)^3}\ \sigma_{\chi\bar\chi\to ij\gamma}v_{\rm rel}\ \left[f_\chi f_{\bar\chi}(1\pm f_i)(1\pm f_j)(1+f_\gamma) - f_i f_j f_\gamma (1\pm f_\chi)(1\pm f_{\bar\chi})\right]$$

Relic Density at NLO

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$$p_\chi + p_{\bar\chi} = p_i + p_j \pm p_\gamma \Rightarrow$$

photon can be arbitrarily soft

$f_\gamma \sim \omega^{-1}$

Maxwell approx. not valid anymore...

...even bigger problem: *T*-dependent IR divergence!

Relic Density

what Really happens at NLO?

$$
\begin{aligned}
C_{\text{NLO}} \sim \int d\Pi_{\chi\bar{\chi}ij} \; f_\chi f_{\bar{\chi}} \; \Big\{ & |\mathcal{M}^{\text{LO}}_{\chi\bar{\chi}\to ij}|^2 + |\mathcal{M}^{\text{NLO}\;T=0}_{\chi\bar{\chi}\to ij}|^2 + \int d\Pi_\gamma |\mathcal{M}_{\chi\bar{\chi}\to ij\gamma}|^2 + \\
& |\mathcal{M}^{\text{NLO}\;T\neq 0}_{\chi\bar{\chi}\to ij}|^2 + \int d\Pi_\gamma \left[f_\gamma \left(|\mathcal{M}_{\chi\bar{\chi}\to ij\gamma}|^2 + |\mathcal{M}_{\chi\bar{\chi}\gamma\to ij}|^2 \right) \right. \\
& \left. - f_i \left(|\mathcal{M}_{\chi\bar{\chi}\to ij\gamma}|^2 + |\mathcal{M}_{\chi\bar{\chi}i\to j\gamma}|^2 \right) - f_j \left(|\mathcal{M}_{\chi\bar{\chi}\to ij\gamma}|^2 + |\mathcal{M}_{\chi\bar{\chi}j\to i\gamma}|^2 \right) \right] \Big\} \\
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Relic Density
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Relic Density

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only this used in NLO literature so far

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thermal 1-loop

photon absorption

photon emission

SM fermions emission

SM fermions absorption

QUESTIONS:

Beneke, Dighera, AH, 1409.3049

1. how the (soft and collinear) IR divergence cancellation happen?

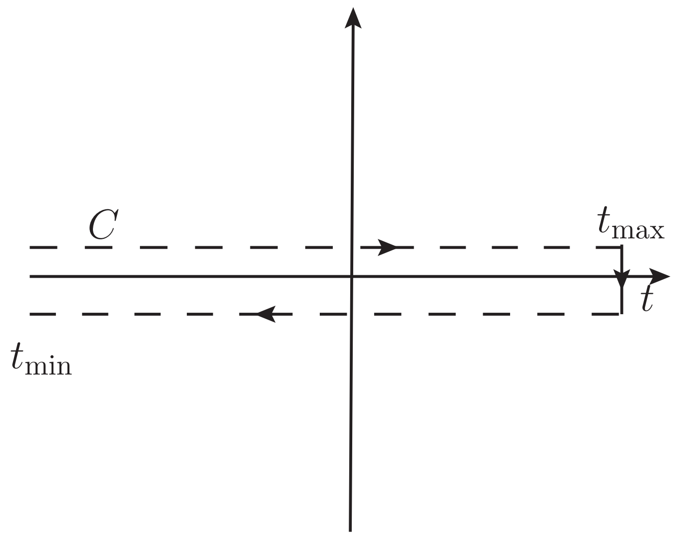
2. does Boltzmann equation itself receive quantum corrections?

3. how large are the remaining finite T corrections?

Program: develop a method for relic density calculation directly from QFT and free from IR problems

framework exists: non-equilibrium thermal field theory

Closed Time Path
FORMALISM



$$i\Delta(x,y) = \langle T_C \phi(x)\phi^\dagger(y)\rangle,$$
$$iS_{\alpha\beta}(x,y) = \langle T_C \psi_\alpha(x)\bar{\psi}_\beta(y)\rangle,$$

contour Green's functions obey Dyson-Schwinger eqs:

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$$(-\partial^2 - m_\phi^2)\Delta^{\lessgtr}(x,y) - \int d^4z \left(\Pi_h(x,z)\Delta^{\lessgtr}(z,y) - \Pi^{\lessgtr}(x,z)\Delta_h(z,y)\right) = \mathcal{C}_\phi,$$
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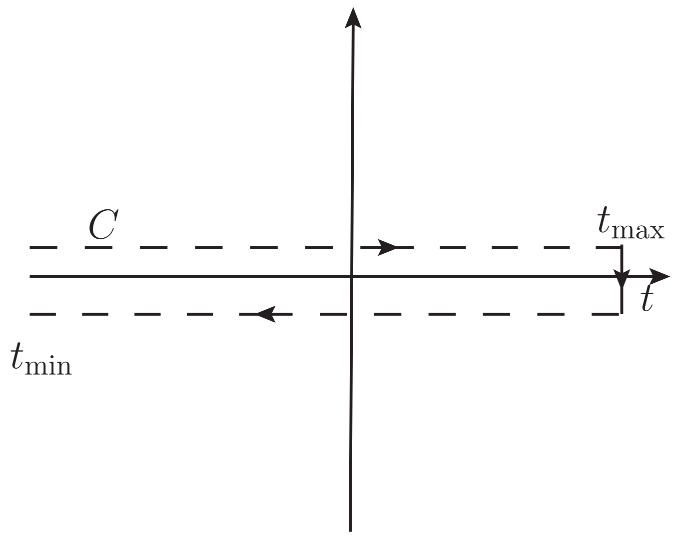
Closed Time Path

FORMALISM



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Closed Time Path
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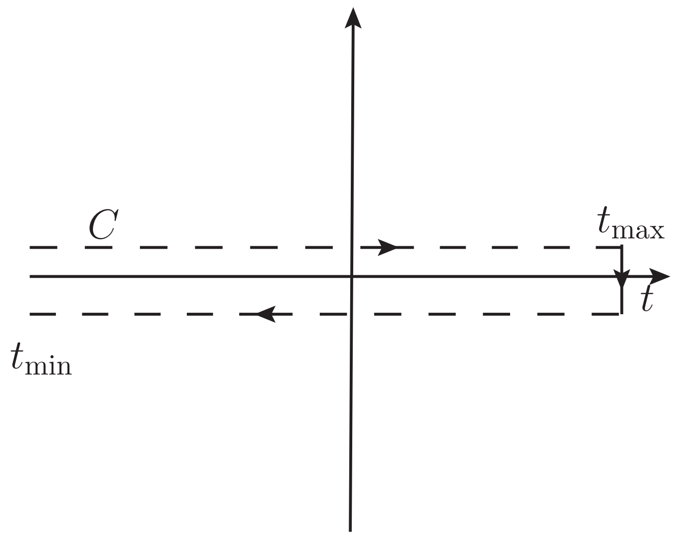

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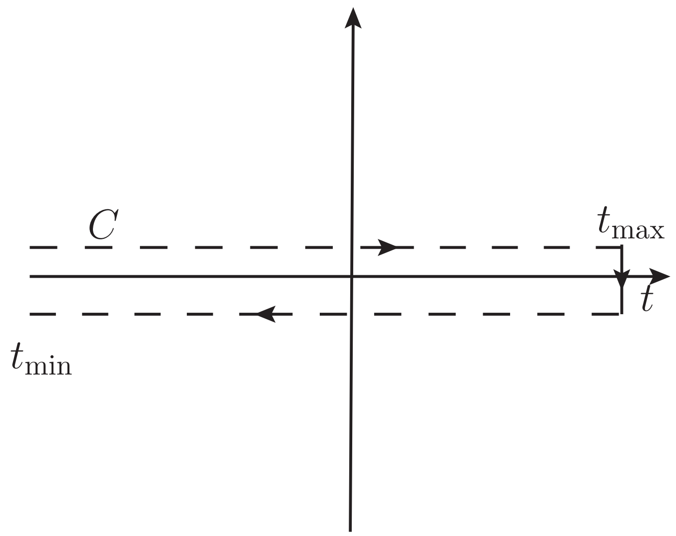
CLOSED TIME PATH
FORMALISM



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CLOSED TIME PATH

FORMALISM: COLLISION TERM

the fermion collision terms is defined as:

propagators

$$\mathcal{C}_\chi = \frac{1}{2}\int d^4z \left(\Sigma^>(x,z)S^<(z,y) - \Sigma^<(x,z)S^>(z,y)\right)$$

self-energies

where the propagators:

thermal part

$$iS^c(p) = \frac{i\left(\not{p}+m\right)}{p^2-m^2+i\eta} - 2\pi\left(\not{p}+m\right)\delta\left(p^2-m^2\right)f\left(p^0\right)$$

$$iS^a(p) = -\frac{i\left(\not{p}+m\right)}{p^2-m^2+i\eta} + 2\pi\left(\not{p}+m\right)\delta\left(p^2-m^2\right)\left(1-f\left(p^0\right)\right)$$

$$iS^>(p) = 2\pi\left(\not{p}+m\right)\delta\left(p^2-m^2\right)\left(1-f\left(p^0\right)\right)$$

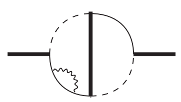
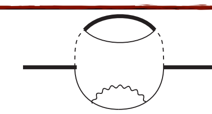
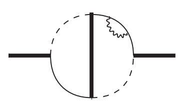
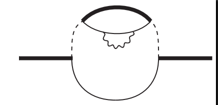
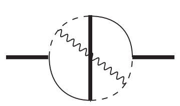
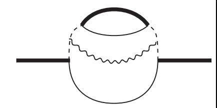
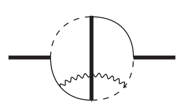
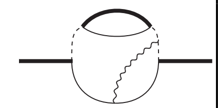
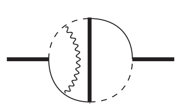
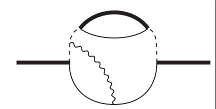
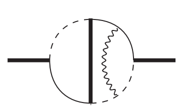
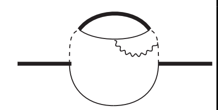
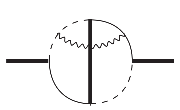
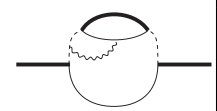
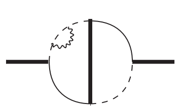
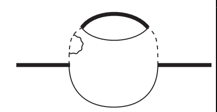
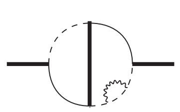
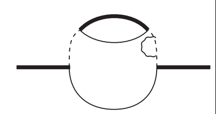
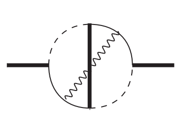
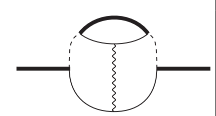
$$iS^<(p) = -2\pi\left(\not{p}+m\right)\delta\left(p^2-m^2\right)f\left(p^0\right)$$

} "cut" propagators

the presence of distribution functions inside propagators $\Rightarrow$ known collision term structure

RESULTS

IR DIVERGENCE CANCELLATION: S-WAVE

The divergent part J_{-1}							
Type A	Real	Virtual	External	Type B	Real	Virtual	External
	$\frac{\alpha(1-2\epsilon^2)}{\pi\epsilon^2}$		$-\frac{\alpha(1-2\epsilon^2)}{\pi\epsilon^2}$		$-\frac{\alpha}{\pi\epsilon^2}$		$\frac{\alpha}{\pi\epsilon^2}$
	$\frac{\alpha(1-2\epsilon^2)}{\pi\epsilon^2}$		$-\frac{\alpha(1-2\epsilon^2)}{\pi\epsilon^2}$		$-\frac{\alpha}{\pi\epsilon^2}$		$\frac{\alpha}{\pi\epsilon^2}$
	0				0		
	0	0			0	0	
	0	0			0	0	
	0	0			0	0	
	0	0			0	0	
		0				0	
		0				0	
	$\frac{2\alpha(1-2\epsilon^2)^2}{\pi\epsilon^2\sqrt{1-4\epsilon^2}}L$	$-\frac{2\alpha(1-2\epsilon^2)^2}{\pi\epsilon^2\sqrt{1-4\epsilon^2}}L$			$\frac{2\alpha(1-2\epsilon^2)}{\pi\epsilon^2\sqrt{1-4\epsilon^2}}L$	$-\frac{2\alpha(1-2\epsilon^2)}{\pi\epsilon^2\sqrt{1-4\epsilon^2}}L$	

→ cancels in **every** row separately

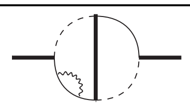
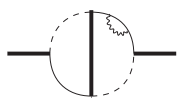
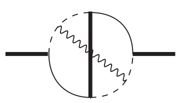
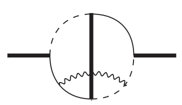
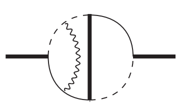
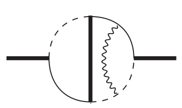
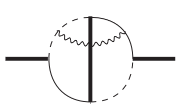
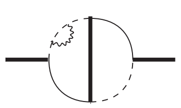
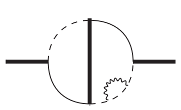
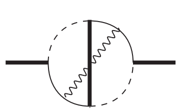
$\Longrightarrow$ every CTP self-energy is IR finite

RESULTS

FINITE T CORRECTION: S-WAVE

factorized $\frac{\pi}{6}\alpha\tau^2\frac{a_{\text{tree}}}{\epsilon^2}$

The finite part J_1

Type A	Real	Virtual	External
	$\frac{2(1-\xi^2)}{D^2D_\xi^2}+\frac{(1-2\epsilon^2)p_1(\epsilon,\xi)}{2D^2D_\xi^2}+\frac{1}{2\sqrt{D}}L$		$\frac{(1-2\epsilon^2)(\xi^2-3D)}{2DD_\xi}-\frac{1}{2\sqrt{D}}L$
	— " —		— " —
	$-\frac{4(1-2\epsilon^2)D}{D_\xi^2}$		
	$-\frac{2(1-2\epsilon^2)\xi^2}{D_\xi^2}-\frac{f_1(\epsilon,\xi)}{\sqrt{D}D_\xi^2}L$	$\frac{2(1-2\epsilon^2)(D-\xi^2)}{D_\xi^2}+\frac{f_1(\epsilon,\xi)}{\sqrt{D}D_\xi^2}L$	
	— " —	— " —	
	— " —	— " —	
	— " —	— " —	
		$-\frac{4(1-2\epsilon^2)D}{D_\xi^2}$	
		— " —	
	$\frac{2(1-2\epsilon^2)p_2(\epsilon,\xi)+(1-\xi^2)^2}{D^2D_\xi^2}+\frac{4f_2(\epsilon,\xi)}{\sqrt{D}D_\xi^2}L$	$\frac{16\epsilon^2(2-3\epsilon^2)-(3-\xi^2)^2}{D_\xi^2}-\frac{4f_2(\epsilon,\xi)}{\sqrt{D}D_\xi^2}L$	

Log terms cancels in **every** row separately

$\Downarrow$

no collinear divergence!

$\xi = \frac{m_\phi}{m_\chi} \gtrsim 1$

$\tau = \frac{T}{m_\chi} \ll 1$

$\epsilon = \frac{m_f}{2m_\chi} \ll \tau$

separate contributions complicated, but when summed up...

$$\Delta a_{\tau^4}^{\epsilon=0} = \frac{8\pi^2\lambda^4\alpha\tau^4}{45}\frac{1}{(1+\xi^2)^4} = \frac{4\pi}{45}\alpha\tau^4\frac{1}{(1+\xi^2)^2}\frac{a_{\text{tree}}}{\epsilon^2}\Big|_{\epsilon=0}$$

strongly suppressed as at kinetic equilibrium $\tau \sim v^2$

THE POWER OF THERMAL OPE

M. Beneke, F. Dighera, AH, 1607.03910

The cross section can be written as the Im part of the forward scattering amplitude:

$$\sigma v_{\rm rel} = \frac{2}{s}\,\mathrm{Im}\left\{(-i)\int d^4x\,\frac{1}{4}\sum_{\rm spin}\langle\bar\chi\chi;T|\,\mathcal{T}\left\{\mathcal{O}_{\rm ann}(0)\,\mathcal{O}^\dagger_{\rm ann}(x)\right\}|\bar\chi\chi;T\rangle\right\}$$

clear separation of soft (thermal effects) and hard (annihilation/decay) modes

$$T \ll m$$

$\Rightarrow$ **Operator Product Expansion**

$$-i\int d^4x\; e^{-ip\cdot x}\,\mathcal{T}\left\{J^\mu_A(0)\,J^{\nu\dagger}_B(x)\right\} = \sum_i C^i_{AB}(p)\cdot\mathcal{O}_i$$

Wilson coeffs. matched at T=0

Possible operators up to dim 4:

$$\mathbb{1}\;,\qquad F^{\alpha\beta}F^{\gamma\delta}\;,\qquad m_f\bar f\,\Gamma f\;,\qquad \bar f\,\Gamma\, iD^\alpha f$$

Matrix elements: LO, $\mathcal{O}(\alpha T^4)$, $\mathcal{O}(\alpha m_f^2 T^2)$, $\mathcal{O}(\alpha T^4)$

No dim 2 operator!

No IR divergence to begin with!

ADVANTAGES OF OPE

- The scaling with T is manifest
- Separation of T=0 and T-dependent contributions
- Significant simplification of the computations
- Clear physics interpretation: at $\mathcal{O}(\alpha\tau^2)$ effects of thermal kinetic energy

Example: muon decay in thermal bath*

Czarnecki et al. '11

$$-i\int d^4x\, e^{-ip\cdot x}\, \mathcal{T}\{J^\mu(0)\, J^{\nu\dagger}(x)\} = C_0^{\mu\nu}\bar{\psi}\psi + C_2^{\mu\nu}\bar{\psi}\frac{i}{2}\sigma_{\alpha\beta}F^{\alpha\beta}\psi + \mathcal{O}(m_\psi^{-3}),$$

*Analogy: semi-leptonic H_b decay in QCD

In the Literature:

OPE in finite temperature - Hatsuda, Koike, Lee '93; Mallik '97; …

Related EFT approach - Biondini, Brambilla, Escobedo, Vairo '13; …

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$$\bar{\psi}\psi = \bar{\psi}\not{v}\psi + \frac{1}{2m_\psi^2}\bar{\psi}\,(iD_\perp)^2\,\psi + \frac{i}{4m_\psi^2}\bar{\psi}\sigma_{\alpha\beta}F^{\alpha\beta}\psi + \mathcal{O}(m_\psi^{-3}),$$

LO $\qquad \mathcal{O}(\alpha\tau^2)$

…and the final correction:

$$\Gamma_T = \Gamma_0\,(1 - K_\psi) + \mathcal{O}(T^3/m_\psi^3).$$

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Conclusions

1. how the (soft and collinear) IR divergence cancellation happen?
   automatic in thermal QFT formalism, cancellation at the level of every CTP self-energy

2. does Boltzmann equation itself receive quantum corrections?
   no, not at NLO

3. how large are the remaining finite T corrections?
   strongly suppressed, of order $\mathcal{O}(\alpha T^4)$

4. the thermal OPE method provides a useful tool and also physics interpretation of the thermal correction

To take home:
complete framework for computations of relic density at NLO w/ thermal effects

Backup Slides

Closed Time Path

path to Boltzmann equation

Kadanoff-Baym $\Longrightarrow$ Boltzmann

$$E\left(\partial_t - H\vec{p}\cdot\nabla_{\vec{p}}\right)f = \mathcal{C}[f].$$

collision term **derived** from thermal QFT

Assumptions:

weak inhomogeneity

gradient expansion

quasi-particle approx.

weak interactions

Justification:

inhomogeneity

plasma excitation momenta

$$\partial \ll k$$

freeze-out happens close to equilibrium

Results

coming back to our example...

every contribution can be written in a form:

$$\int_0^\infty d\omega\, f_\gamma(\omega)\, \boxed{S(\omega, e_\chi, \epsilon, \xi)} \qquad f_\gamma(\omega) = \frac{1}{1-e^{\omega/T}}$$

photon energy

expand in ω

$$S = \sum_{i=-1}^{\infty} s_n \omega^n$$

note:

$$J_n \equiv \int_0^\infty f_B(\omega)\omega^n d\omega = \begin{cases} \text{div} & n \leq 0 \\ \sim \tau^{n+1} & n > 0 \end{cases}$$

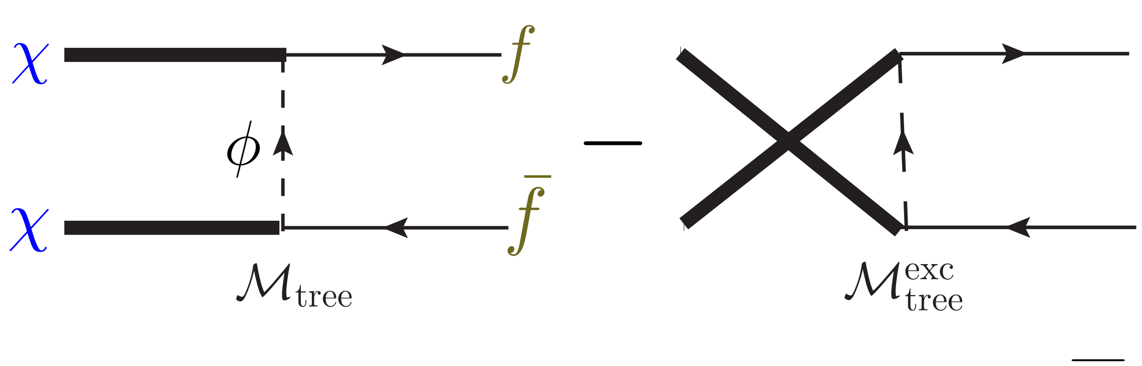
IR divergence in separate terms: $J_{-1} \leftrightarrow T = 0$ soft div, $J_0 \leftrightarrow T = 0$ soft eikonal

finite T corrections: $J_1 \leftrightarrow \mathcal{O}(\tau^2)$ ...

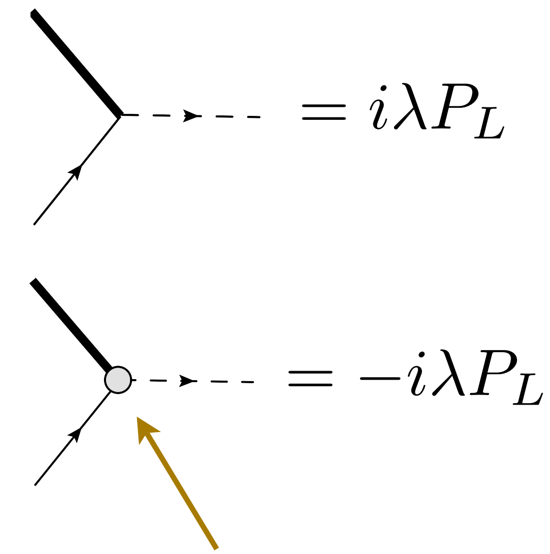
COLLISION TERM
EXAMPLE

Bino-like DM: χ Majorana fermion, SM singlet

annihilation process at tree level:



vertices (2 types):



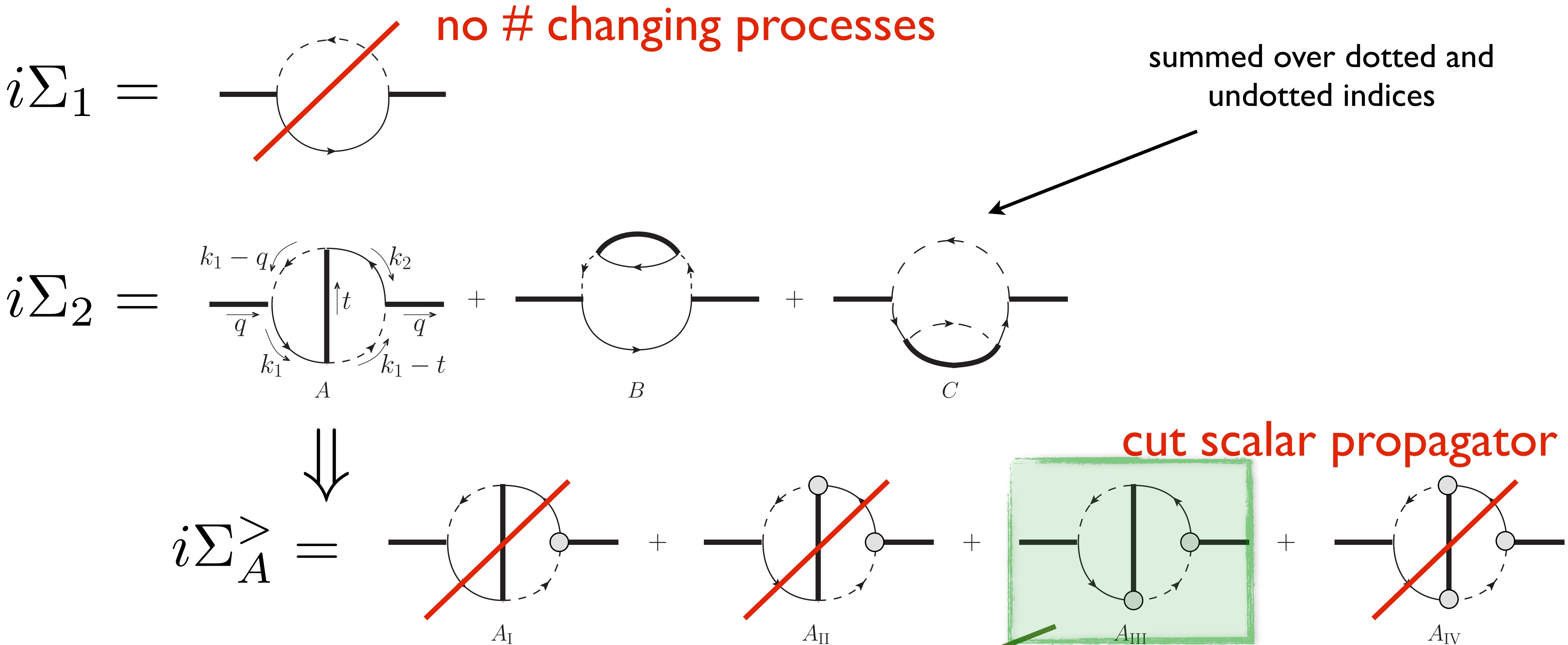
scale hierarchy: $\boxed{m_\phi \gtrsim m_\chi} \gg T \gg \boxed{m_f}$

no thermal contributions

effectively massless

rescaled variables: $\tau = \dfrac{T}{m_\chi} \ll 1$ $\quad \epsilon = \dfrac{m_f}{2m_\chi} \ll \tau$ $\quad \xi = \dfrac{m_\phi}{m_\chi} \gtrsim 1$

Collision Term
computation



$$\Sigma^{>}_{A_{\rm III}}(q)\,S^{<}(q) = -\lambda^4 \int \frac{d^4t}{(2\pi)^4}\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}(2\pi)^4\,\delta(q+t-k_1-k_2)$$

$$\underbrace{i\Delta^{11}(k_1-q)\,i\Delta^{22}(k_1-t)}_{\equiv\mathcal{S}}\underbrace{P_{\rm R}iS^{21}(k_2)\,P_{\rm L}iS^{12}(t)\,P_{\rm L}iS^{21}(k_1)\,P_{\rm R}iS^{12}(q)}_{\equiv\mathcal{F}}$$

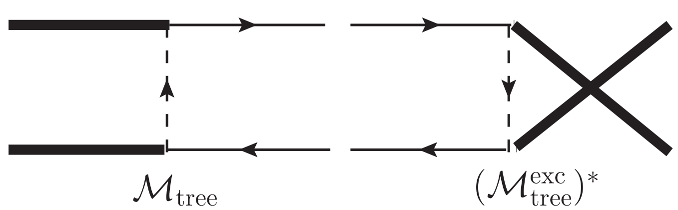
Collision Term
MATCHING

after inserting the propagators:

$$\Sigma^{>}_{A_{\rm III}}(q)\, S^{<}(q) = \frac{1}{2E_{\chi_1}} (2\pi)\,\delta\left(q^0 - E_{\chi_1}\right) \int \frac{d^4t}{(2\pi)^3\, 2E_{\chi_2}} \delta\left(t^0 - E_{\chi_2}\right) \times$$

$$\int \frac{d^3\vec{k}_1}{(2\pi)^3\, 2E_{f_1}} \frac{d^3\vec{k}_2}{(2\pi)^3\, 2E_{f_2}} (2\pi)^4\, \delta\left(q + t - k_1 - k_2\right) |\mathcal{M}_A|^2 \left[f_\chi(q)\, f_\chi(t) \left(1 - f_f^{\rm eq}\left(k_1^0\right)\right)\left(1 - f_f^{\rm eq}\left(k_2^0\right)\right)\right]$$

$\Rightarrow$ one indeed recovers the known collision term and

$|\mathcal{M}_A|^2 =$ 

(part of) tree level $|\mathcal{M}|^2$

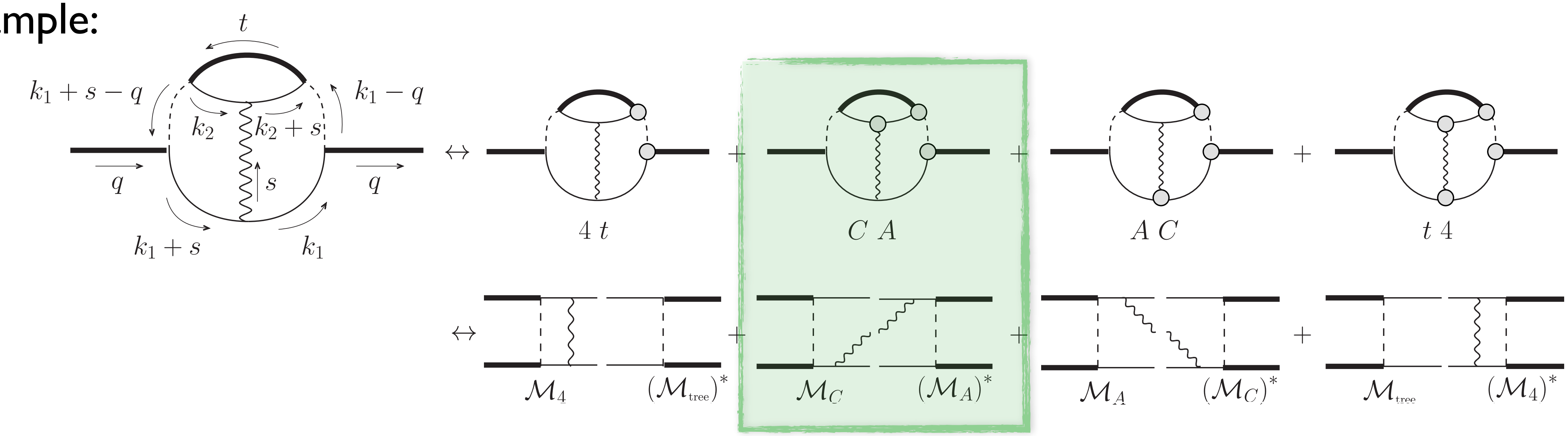
repeating the same for B type diagrams the bottom line:

$i\Sigma^{>} \leftrightarrow$ tree level annihilation contribution to the collision term

COLLISION TERM
MATCHING AT NLO

$i\Sigma_3 =$ 20 self-energy diagrams

example:



$$\Sigma^{>}_{\text{CA}}(q)\, S^{<}(q) = \frac{1}{2E_{\chi_1}} (2\pi)\, \delta\left(q^0 - E_{\chi_1}\right) \int \frac{d^4 t}{(2\pi)^3\, 2E_{\chi_2}} \delta\left(t^0 - E_{\chi_2}\right)$$

$$\int \frac{d^3\vec{k}_1}{(2\pi)^3\, 2E_{f_1}} \frac{d^3\vec{k}_2}{(2\pi)^3\, 2E_{f_2}} \frac{d^3\vec{s}}{(2\pi)^3\, 2E_{\gamma}} (2\pi)^4\, \delta\left(q + t - k_1 - k_2 - s\right)$$

$$\mathcal{M}_C\left(\mathcal{M}_A\right)^* \left[f_\chi(q)\, f_\chi(t) \left(1 - f_f^{\text{eq}}\left(k_1^0\right)\right)\left(1 - f_f^{\text{eq}}\left(k_2^0\right)\right)\left(1 + f_\gamma^{\text{eq}}\left(s^0\right)\right)\right]$$

$\Rightarrow$ at NLO thermal effects do **not** change the collision therm structure