(More) exceptions
in the calculations of Relic Abundances

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based on: **T. Binder, T. Bringmann, M. Gustafsson and AH,** 1706.07433

M. Beneke, F. Dighera, AH, 1409.3049, 1607.03910

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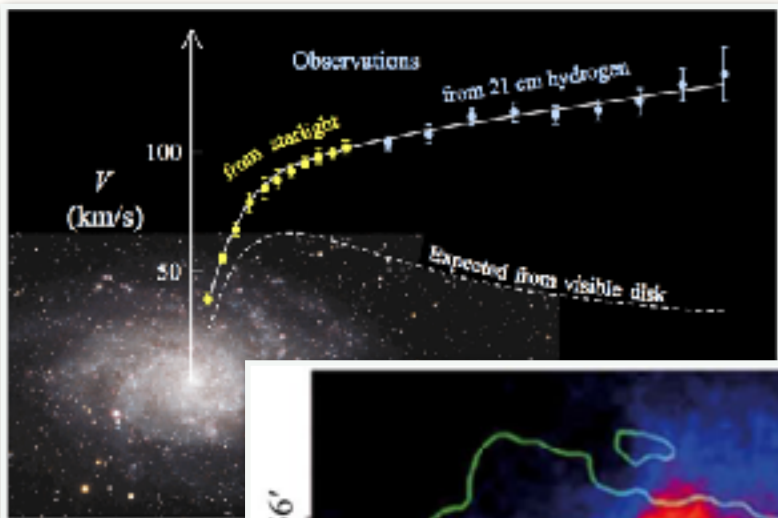
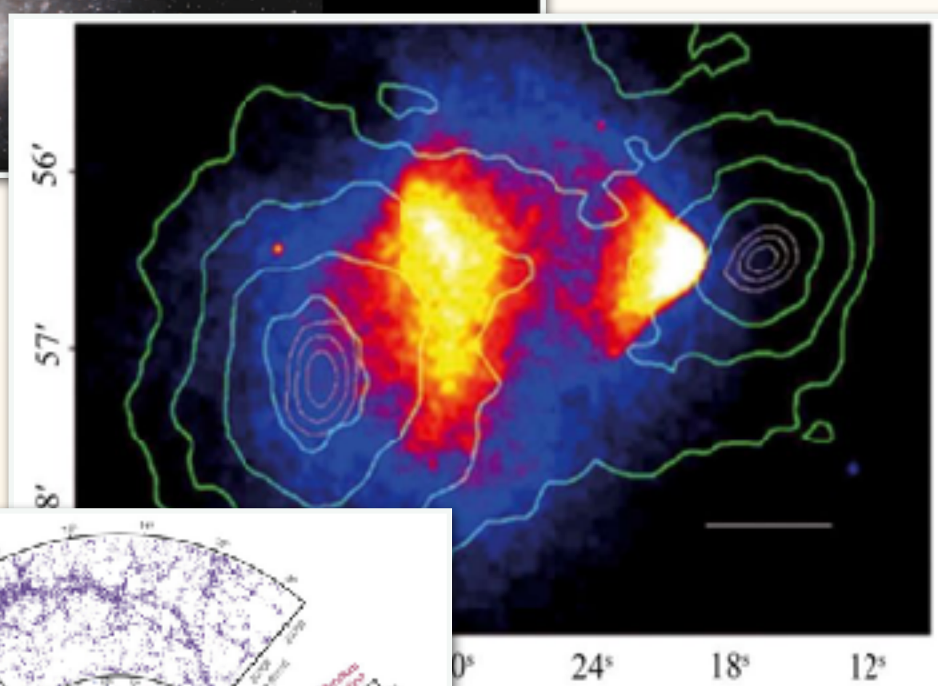
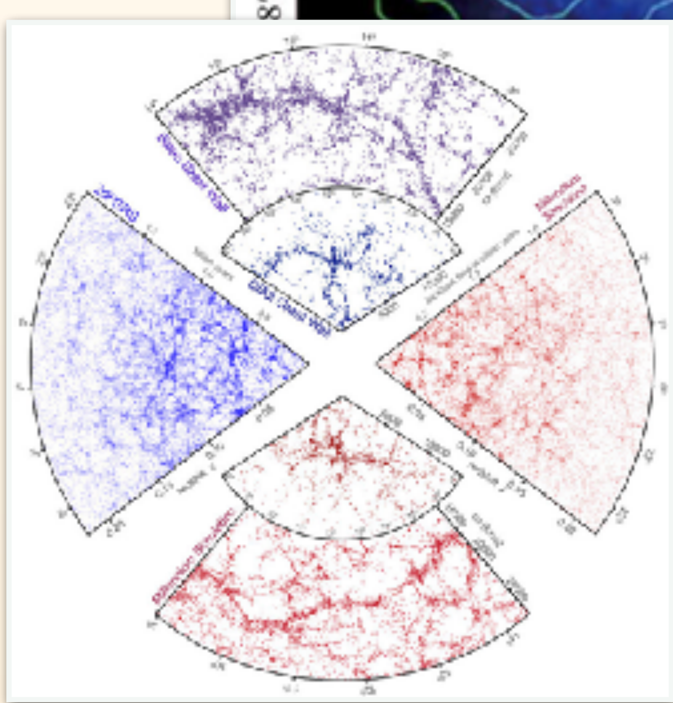
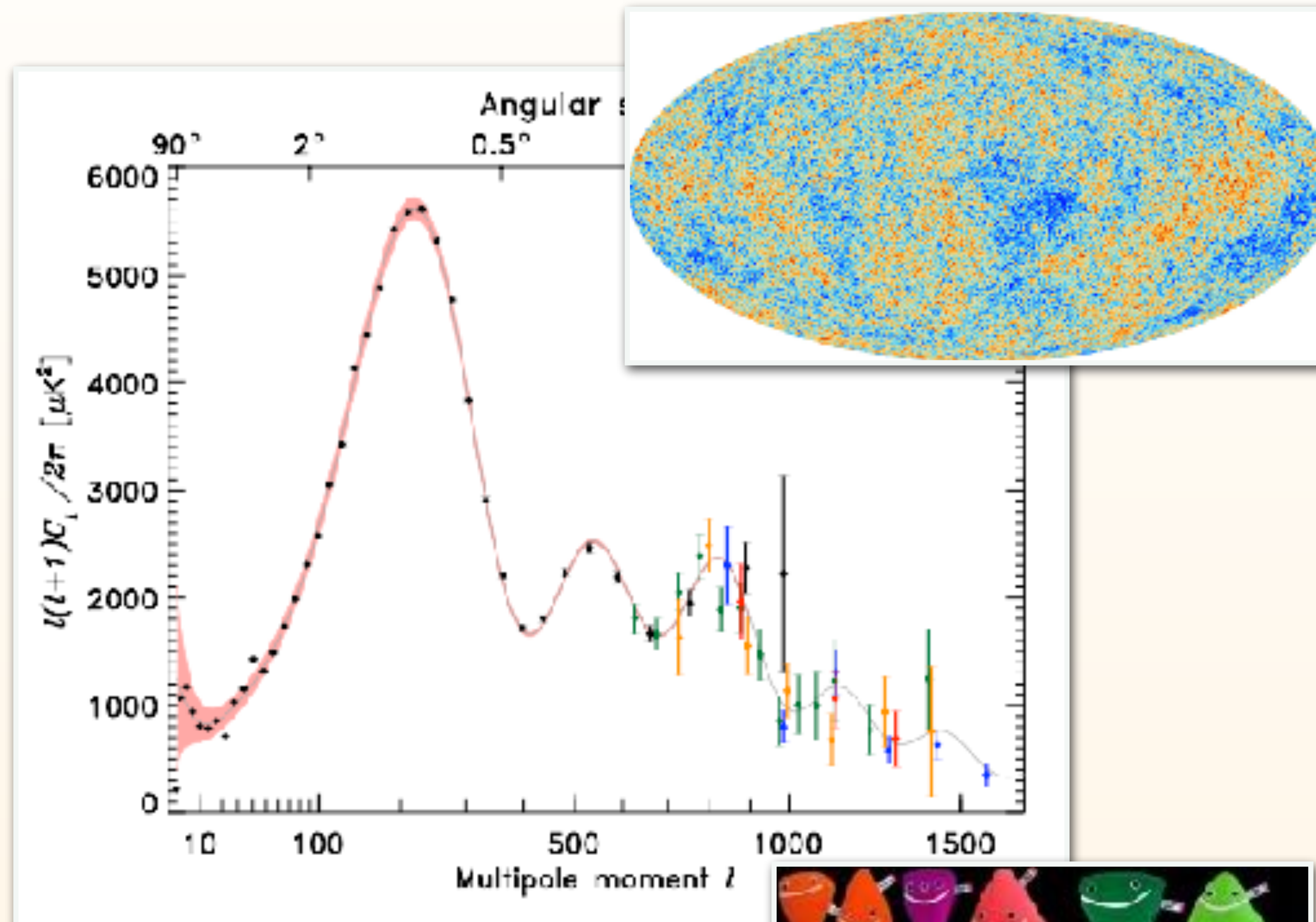
Dark Matter

is everywhere!









$\Longrightarrow$ Evidence on all scales!

The Origin of Dark Matter

Dark matter could be created in many different ways…

…but <u>every</u> massive particle with not-too-weak interactions with the SM will be produced thermally, with relic abundance:

Lee, Weinberg '77; + others

$$\Omega_\chi h^2 \approx 0.1 \, \frac{3 \times 10^{-26} \mathrm{cm}^3 \mathrm{s}^{-1}}{\langle \sigma v \rangle}$$

It is very natural to expect that this mechanism is responsible for the origin of all of dark matter

…but even if not, it still is present nevertheless and it's important to be able to correctly determine thermal population abundance

Historical Prelude

Three Exceptions

Griest & Seckel '91

1. Co-annihilations

if more than one state share a conserved quantum number making DM stable

$$\langle \sigma_{\rm eff} v \rangle = \sum_{ij} \langle \sigma_{ij} v_{ij} \rangle \frac{n_i^{\rm eq} n_j^{\rm eq}}{n_{\rm eq}^2}$$

with: $\sigma_{ij} = \sum_X \sigma(\chi_i \chi_j \to X)$

e.g., SUSY

2. Annihilation to forbidden channels

if DM is slightly below mass threshold for annihilation $\Longrightarrow$ „forbidden" channel can still be accessible in thermal bath

recent e.g., 1505.07107

3. Annihilation near poles

expansion in velocity (s-wave, p-wave, etc.) not safe

(more historical issue: these days most people use numerical codes)

THERMAL RELIC DENSITY

MODERN "EXCEPTIONS"

1. Non-standard cosmology

   many works… very recent e.g., D'Eramo, Fernandez, Profumo '17

2. Bound State Formation

   recent e.g., Petraki at al. '15, '16; An et al. '15, '16; Cirelli et al. '16; …

3. $3 \rightarrow 2$ and $4 \rightarrow 2$ annihilation

   e.g., D'Agnolo, Ruderman '15; Cline at al. '17; Choi at al. '17; …

4. Second era of annihilation

   Feng et al. '10; Bringmann et al. '12; …

5. Semi-annihilation

   D'Eramo, Thaler '10; …

6. Cannibalization

   e.g., Kuflik et al. '15; Pappadopulo et al. '16; …

7. …

In other words: whenever studying non-minimal scenarios "exceptions" appear — but most of them not affect the foundations of modern calculations

OUTLINE

1. Introduction
   - standard approach to thermal relic density

2. Exception ***n***
   - early kinetic decoupling with
   - velocity dependent annihilation

3. Exception ***n+1***
   - NLO effects at finite temperature

4. Summary

Thermal Relic Density

STANDARD APPROACH

T		
	$\Gamma_{\mathrm{ann}} > H$	DM in equilibrium
	$\Gamma_{\mathrm{ann}} \sim H$	$\boxed{\text{chemical decoupling}}$
	$\Gamma_{\mathrm{ann}} < H$	freeze-out

time

non-equilibrium

time evolution of $f_\chi(p)$ in kinetic theory:

$$\boxed{E\left(\partial_t - H\vec{p}\cdot\nabla_{\vec{p}}\right)} f_\chi = \mathcal{C}[f_\chi] \quad \Rightarrow \quad \frac{dn_\chi}{dt} + 3Hn_\chi = C$$

Liouville operator in FRW background

the collision term

integrated

assumptions for using Boltzmann eq: classical limit, molecular chaos,...

(for derivation from thermal QFT… see second part of the talk)

Thermal Relic Density

The collision term

for $2 \leftrightarrow 2$ CP invariant process:

$$C_{\rm LO} = -h_\chi^2 \int \frac{d^3\vec{p}_\chi}{(2\pi)^3}\frac{d^3\vec{p}_{\bar\chi}}{(2\pi)^3}\,\sigma_{\chi\bar\chi\to ij}v_{\rm rel}\,\left[f_\chi f_{\bar\chi}(1\pm f_i)(1\pm f_j) - f_i f_j (1\pm f_\chi)(1\pm f_{\bar\chi})\right]$$

assuming kinetic equilibrium at chemical decoupling: $f_\chi \sim a(\mu) f_\chi^{\rm eq}$

$$C_{\rm LO} = -\langle\sigma_{\chi\bar\chi\to ij}v_{\rm rel}\rangle^{\rm eq}\left(n_\chi n_{\bar\chi} - n_\chi^{\rm eq} n_{\bar\chi}^{\rm eq}\right)$$

where the thermally averaged cross section:

$$\langle\sigma_{\chi\bar\chi\to ij}v_{\rm rel}\rangle^{\rm eq} = -\frac{h_\chi^2}{n_\chi^{\rm eq} n_{\bar\chi}^{\rm eq}}\int \frac{d^3\vec{p}_\chi}{(2\pi)^3}\frac{d^3\vec{p}_{\bar\chi}}{(2\pi)^3}\,\sigma_{\chi\bar\chi\to ij}v_{\rm rel}\, f_\chi^{\rm eq} f_{\bar\chi}^{\rm eq}$$

Thermal Relic Density

Boltzmann eq.

$$\frac{dn_\chi}{dt} + 3Hn_\chi = -\langle \sigma_{\chi\bar{\chi}\to ij}\sigma_{\rm rel}\rangle^{\rm eq}\left(n_\chi n_{\bar{\chi}} - n_\chi^{\rm eq} n_{\bar{\chi}}^{\rm eq}\right)$$

Re-written for the comoving number density:

$$\frac{dY}{dx} = \sqrt{\frac{g_* \pi m_\chi^2}{45G}}\,\frac{\langle \sigma_{\chi\bar{\chi}\to ij}\sigma_{\rm rel}\rangle^{\rm eq}}{x^2}\left(Y^2 - Y_{\rm eq}^2\right)$$

$$\lim_{x\to 0} Y = Y_{\rm eq} \qquad \lim_{x\to\infty} Y = \text{const}$$

Recipe:
compute annihilation cross-section,
take a thermal bath average,
throw it into BE… and voilà

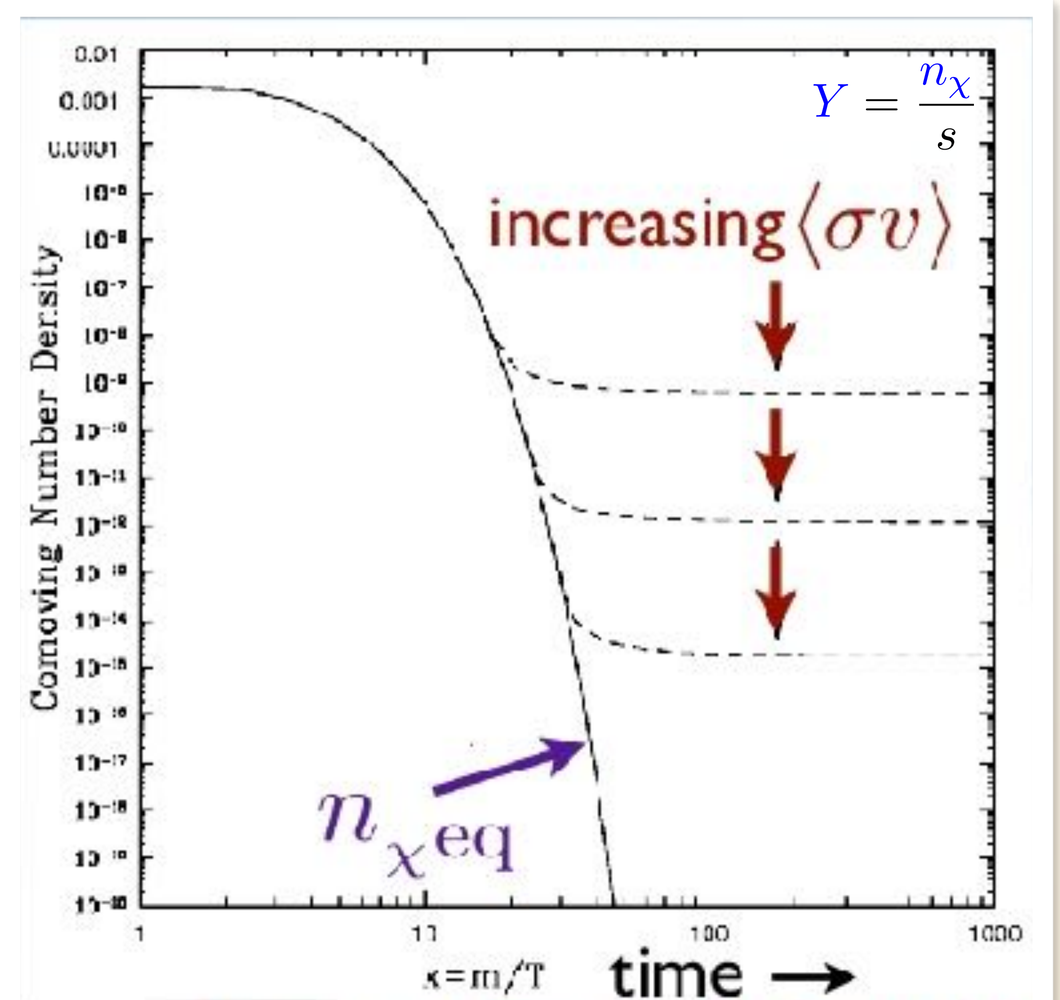
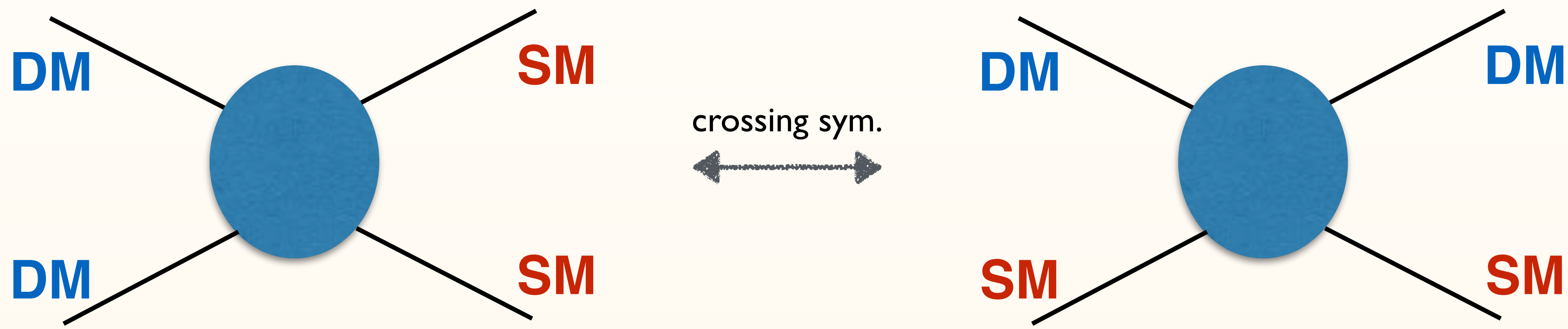


Fig.: Jungman, Kamionkowski & Griest, PR'96

Freeze-out *vs.* Decoupling

annihilation | (elastic) scattering



$$\sum_{\text{spins}} \left|\mathcal{M}^{\text{pair}}\right|^2 = F(p_1, p_2, p_1', p_2') \quad \sim \quad \sum_{\text{spins}} \left|\mathcal{M}^{\text{scatt}}\right|^2 = F(k, -k', p', -p)$$

Boltzmann suppression of DM vs. SM $\Rightarrow$ scatterings <u>typically</u> more frequent

dark matter frozen-out but <u>typically</u> still kinetically coupled to the plasma

$$\boxed{\tau_{\text{r}}(T_{\text{kd}}) \equiv N_{\text{coll}}/\Gamma_{\text{el}} \sim H^{-1}(T_{\text{kd}})}$$

Schmid, Schwarz, Widern '99; Green, Hofmann, Schwarz '05

Two consequences:

1. During freeze-out (chemical decoupling) <u>typically</u>: $f_\chi \sim a(\mu) f_\chi^{\text{eq}}$
2. If kinetic decoupling much, much later: possible impact on the matter power spectrum

i.e. kinetic decoupling can have observable consequences and affect e.g. missing satellites problem

see e.g. ,Bringmann, Ihle, Karsten, Walia '16

Exception *n*:

Early kinetic decoupling

A PITFALL IN A NUTSHELL

T ↑

$\Gamma_{\text{ann}} > H$ — DM in equilibrium

$\Gamma_{\text{ann}} \sim H$ — chemical decoupling

$\Gamma_{\text{ann}} < H$ — freeze-out

time ↓

can it happen?

typically

kinetic decoupling

If KD happens around CD ⟶ what would be the relic density?

↓

how to even compute that? $\Rightarrow$ need for refined treatment of solving the Boltzmann eq.

assuming kinetic equilibrium at chemical decoupling: $f_\chi \sim a(\mu) f_\chi^{\text{eq}}$

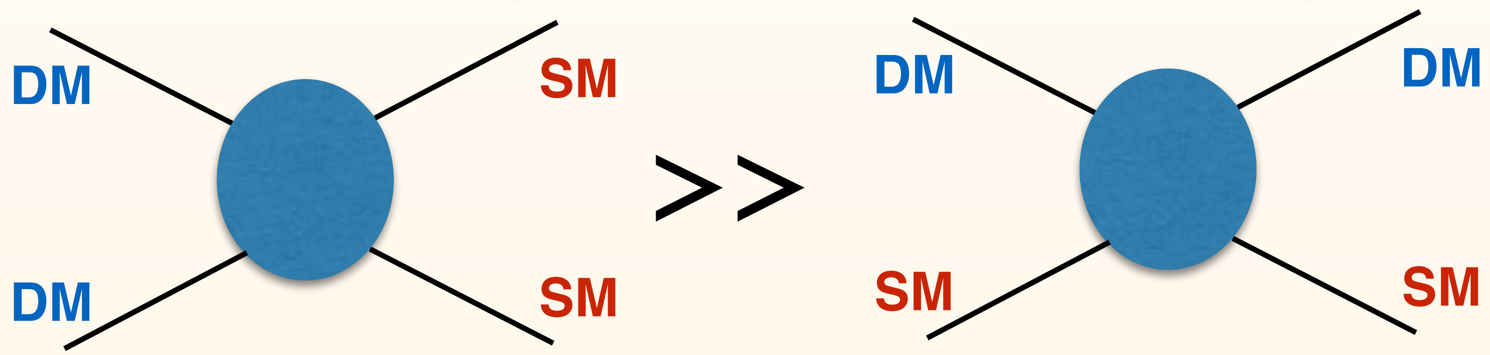
$$C_{\text{LO}} = -\langle \sigma_{\chi\bar{\chi}\to ij} v_{\text{rel}} \rangle^{\text{eq}} \left(n_\chi n_{\bar{\chi}} - n_\chi^{\text{eq}} n_{\bar{\chi}}^{\text{eq}} \right)$$

Early Kinetic Decoupling?

A necessary and sufficient condition: scatterings weaker than annihilation

i.e. rates around freeze-out: $H \sim \Gamma_{\rm ann} \gtrsim \Gamma_{\rm el}$

Possibilities:

A) 

e.g., resonant annihilation

see also Duch, Grządkowski '17

B) Boltzmann suppression of SM as strong as for DM

e.g., below threshold annihilation (forbidden-like DM)

C) Scatterings and annihilation have different structure

e.g., semi-annihilation, 3 to 2 models,…

How to describe KD?

All information is in full BE:

both about chemical ("normalization") and kinetic ("shape") equilibrium/decoupling

$$E\left(\partial_t - H\vec{p}\cdot\nabla_{\vec{p}}\right) f_\chi = \mathcal{C}[f_\chi]$$

contains both scatterings and annihilation

Two possible approaches:

solve numerically for full $f_\chi(p)$

have insight on the distribution

no constraining assumptions

numerically challenging

typically overkill

consider system of equations for moments of $f_\chi(p)$

0-th moment: n_χ

2-nd moment: T_χ

$\ldots$

partially analytic/much easier numerically

manifestly captures all of the relevant physics

finite range of validity

no insight on the distribution

SCATTERING

The elastic scattering collision term:

$$C_{\rm el} = \frac{1}{2g_\chi}\int\frac{d^3k}{(2\pi)^3 2\omega}\int\frac{d^3\tilde{k}}{(2\pi)^3 2\tilde{\omega}}\int\frac{d^3\tilde{p}}{(2\pi)^3 2\tilde{E}}$$
$$\times(2\pi)^4\delta^{(4)}(\tilde{p}+\tilde{k}-p-k)|\mathcal{M}|^2_{\chi f\leftrightarrow\chi f}$$
$$\times\left[(1\mp g^{\pm})(\omega)\, g^{\pm}(\tilde{\omega}) f_\chi(\tilde{\mathbf{p}}) - (\omega\leftrightarrow\tilde{\omega},\mathbf{p}\leftrightarrow\tilde{\mathbf{p}})\right]$$

equilibrium functions for SM particles

Expanding in **NR** and small **momentum transfer**:

Bringmann, Hofmann '06

$$C_{\rm el} \simeq \frac{m_\chi}{2}\gamma(T)\left[T m_\chi \partial_p^2 + \left(p + 2T\frac{m_\chi}{p}\right)\partial_p + 3\right] f_\chi$$

More generally, Fokker-Planck scattering operator (relativistic, but still small **momentum transfer**):

Binder et al. '16

$$C_{\rm el} \simeq \frac{E}{2}\nabla_{\mathbf{p}}\cdot\left[\gamma(T,\mathbf{p})\left(ET\nabla_{\mathbf{p}} + \mathbf{p}\right) f_\chi\right]$$

<u>Semi-relativistic</u>: assume that scattering $\gamma(T,\mathbf{p})$ is momentum independent

Kinetic Decoupling 101

DM temperature Definition:

$$T_\chi \equiv \frac{g_\chi}{3m_\chi n_\chi}\int \frac{\mathrm{d}^3p}{(2\pi)^3}p^2 f_\chi(p) \qquad y \equiv \frac{m_\chi T_\chi}{s^{2/3}}$$

actually: normalized average NR energy - equals temperature at equilibrium

First take <u>late KD scenario</u> and consider only temperature evolution - i.e. leave out feedback **on/from** changing number density:

then 2nd moment of full BE (up to terms p^2/m_χ^2) gives:

$$\boxed{\frac{y'}{y} = -\cancel{\frac{Y'}{Y}\left(1-\frac{\langle\sigma v_{\rm rel}\rangle_2}{\langle\sigma v_{\rm rel}\rangle}\right)} - \left(1-\frac{x}{3}\frac{g'_{*S}}{g_{*S}}\right)\frac{2m_\chi c(T)}{Hx}\left(1-\frac{y_{\rm eq}}{y}\right)}$$

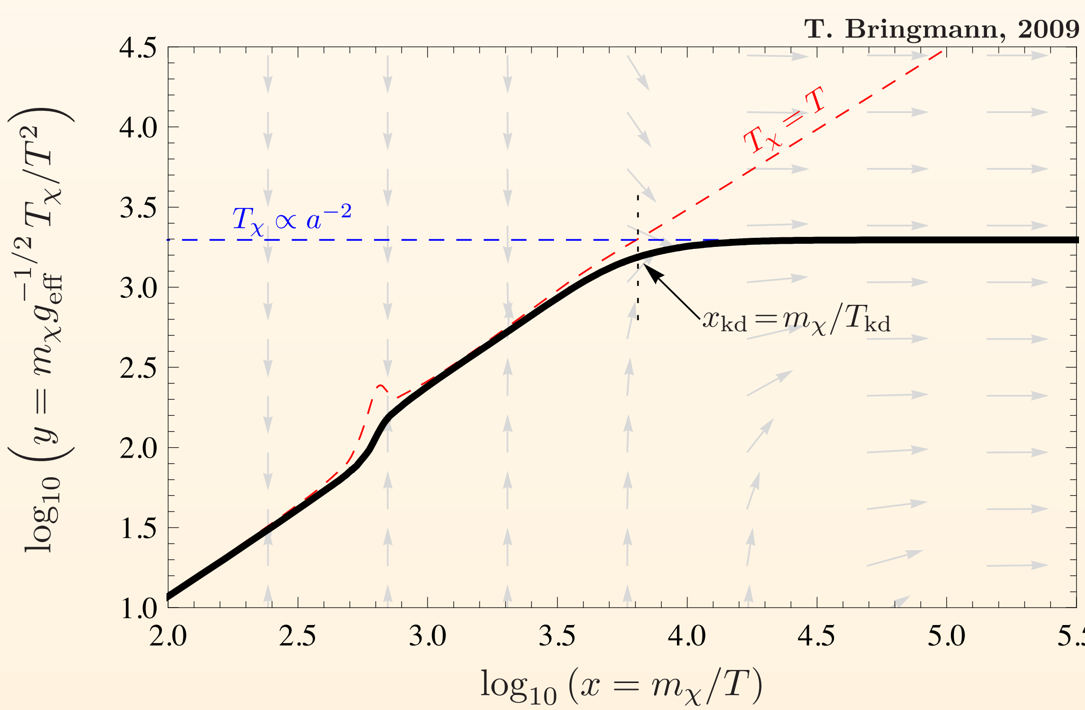
where:

$$\langle\sigma v_{\rm rel}\rangle_2 \equiv \frac{g_\chi^2}{3Tm_\chi n_\chi^2}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3\tilde p}{(2\pi)^3}\, p^2 v_{\rm rel}\sigma_{\bar\chi\chi\to\bar X X} f(E)f(\tilde E)$$

impact of annihilation

$$c(T) = \frac{1}{12(2\pi)^3 m_\chi^4 T}\sum_X\int dk\, k^5\omega^{-1}\, g^\pm\left(1\mp g^\pm\right)\int_{-4k^2}^{0}(-t)\frac{1}{8k^4}|\mathcal{M}_{\rm el}|^2$$

impact of elastic scatterings



ONE STEP FURTHER…

Now consider general KD scenario, i.e. coupled temperature and number density evolution:

annihilation and production thermal averages done at different T — feedback of modified y evolution

$$\frac{Y'}{Y} = -\frac{1-\frac{x}{3}\frac{g'_{*S}}{g_{*S}}}{Hx} sY \left(\langle \sigma v_{\rm rel} \rangle|_{x=m_\chi^2/(s^{2/3}y)} - \frac{Y_{\rm eq}^2}{Y^2} \langle \sigma v_{\rm rel} \rangle|_x \right)$$

$$\frac{y'}{y} = -\frac{1-\frac{x}{3}\frac{g'_{*S}}{g_{*S}}}{Hx} \left[2m_\chi c(T) \left(1 - \frac{y_{\rm eq}}{y}\right) - sY \left(\left(\langle \sigma v_{\rm rel} \rangle - \langle \sigma v_{\rm rel} \rangle_2 \right)_{x=m_\chi^2/(s^{2/3}y)} - \frac{Y_{\rm eq}^2}{Y^2} \left(\langle \sigma v_{\rm rel} \rangle - \frac{y_{\rm eq}}{y} \langle \sigma v_{\rm rel} \rangle_2 \right)_x \right) \right]$$

$$+ \frac{1-\frac{x}{3}\frac{g'_{*S}}{g_{*S}}}{3m_\chi} \langle p^4/E^3 \rangle_{x=m_\chi^2/(s^{2/3}y)}$$

"relativistic" term

$$T_\chi \equiv \frac{g_\chi}{3n_\chi} \int \frac{{\rm d}^3p}{(2\pi)^3} \frac{p^2}{E} f_\chi(p)$$

elastic scatterings term

impact of annihilation

These equations still assume the equilibrium shape of $f_\chi(p)$ — but with variant temperature

or more accurately: that the thermal averages computed with true non-equilibrium distributions don't differ much from the above ones

Numerical approach

… or one can just solve full phase space Boltzmann eq.

$$
\begin{aligned}
\partial_x f_\chi(x,q) \;=\;& \frac{m_\chi^3}{\tilde{H}x^4}\frac{g_{\bar\chi}}{2\pi^2}\int d\tilde{q}\,\tilde{q}^2\,\frac{1}{2}\int d\cos\theta\; v_{\rm Møl}\sigma_{\bar\chi\chi\to\bar f f} \\
& \times\left[f_{\chi,\rm eq}(q)f_{\chi,\rm eq}(\tilde q) - f_\chi(q)f_\chi(\tilde q)\right] \\
+\;& \frac{2m_\chi c(T)}{2\tilde{H}x}\left[x_q\partial_q^2 + \left(q + \frac{2x_q}{q} + \frac{q}{x_q}\right)\partial_q + 3\right] f_\chi \\
+\;& \tilde{g}\frac{q}{x}\partial_q f_\chi,
\end{aligned}
$$

fully general

expanded in NR and small momentum transfer (semi-relativistic!)

discretization, ~1000 steps

$$
\begin{aligned}
\partial_x f_i \;=\;& \\
& \frac{m_\chi^3}{\tilde{H}x^4}\frac{g_{\bar\chi}}{2\pi^2}\sum_{j=1}^{N-1}\frac{\Delta\tilde{q}_j}{2}\Big[\tilde{q}_j^2\,\langle v_{\rm Møl}\sigma_{\bar\chi\chi\to\bar f f}\rangle^\theta_{i,j}\left(f_i^{\rm eq}f_j^{\rm eq} - f_i f_j\right) \\
+\;& \tilde{q}_{j+1}^2\,\langle v_{\rm Møl}\sigma_{\bar\chi\chi\to\bar f f}\rangle^\theta_{i,j+1}\left(f_i^{\rm eq}f_{j+1}^{\rm eq} - f_i f_{j+1}\right)\Big] \\
+\;& \frac{2m_\chi c(T)}{2\tilde{H}x}\left[x_{q,i}\partial_q^2 + \left(q_i + \frac{2x_{q,i}}{q_i} + \frac{q_i}{x_{q,i}}\right)\partial_q + 3\right] f_i \\
+\;& \tilde{g}\frac{q_i}{x}\partial_q f_i,
\end{aligned}
$$

Solved numerically with MatLab

Note:

can be extended to e.g. self-scatterings

very stiff, care needed with numerics

Example:

Scalar Signlet DM

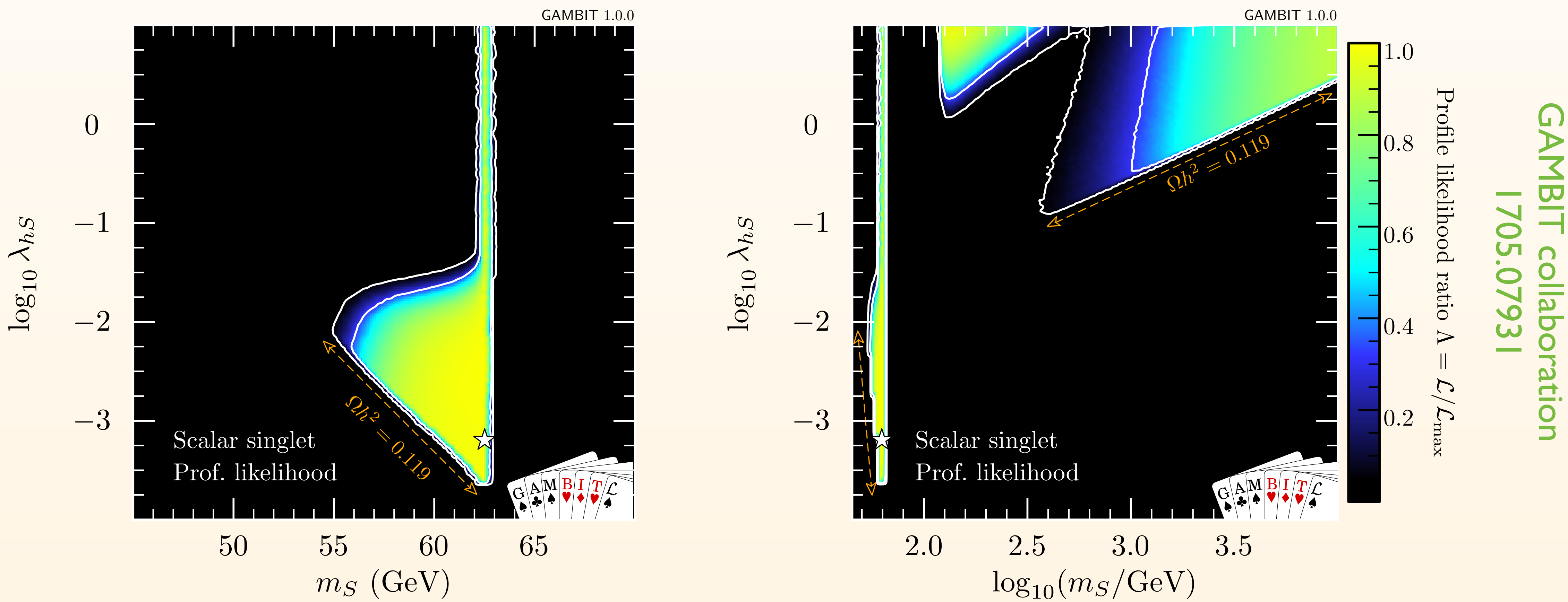
SCALAR SINGLET DM

VERY SHORT INTRODUCTION

To the SM Lagrangian add one singlet scalar field *S* with interactions with the Higgs:

$$\mathcal{L}_S = \frac{1}{2}\partial_\mu S \partial^\mu S - \frac{1}{2}\mu_S^2 S^2 - \frac{1}{2}\lambda_s S^2 |H|^2$$

$$m_s = \sqrt{\mu_S^2 + \frac{1}{2}\lambda_s v_0^2}$$



Most of the parameter space excluded, but… even such a simple model is hard to kill

SCALAR SINGLET DM

ANNIHILATION VS. SCATTERINGS

$$\sigma v_{\rm rel} = \frac{2\lambda_s^2 v_0^2}{\sqrt{s}} |D_h(s)|^2 \Gamma_h(\sqrt{s})$$

tabulated Higgs width

with:

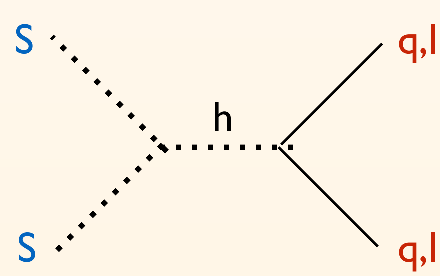
$$|D_h(s)|^2 \equiv \frac{1}{(s-m_h^2)^2 + m_h^2 \Gamma_h^2(m_h)}.$$

$$\left\langle |\mathcal{M}|^2 \right\rangle_t = \sum_f \frac{N_f \lambda_S^2 m_f^2}{8k^4} \Big[\frac{2k_{\rm cm}^2 - 2m_f^2 + m_h^2}{1 + m_h^2/(4k_{\rm cm}^2)} - \left(m_h^2 - 2m_f^2\right) \log\left(1 + 4k_{\rm cm}^2/m_h^2\right) \Big].$$

Hierarchical Yukawa couplings: strongest coupling to more Boltzmann suppressed quarks/leptons

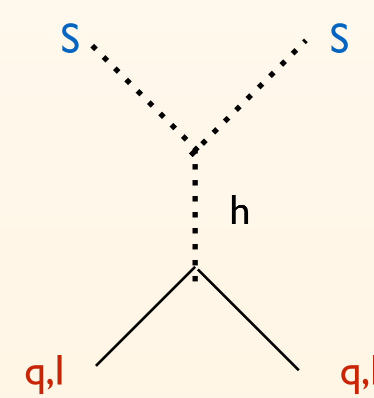
Annihilation processes:

resonant



El. scattering processes:

non-resonant



Freeze-out at few GeV $\longrightarrow$ what is the <u>abundance of heavy quarks</u> in QCD plasma?

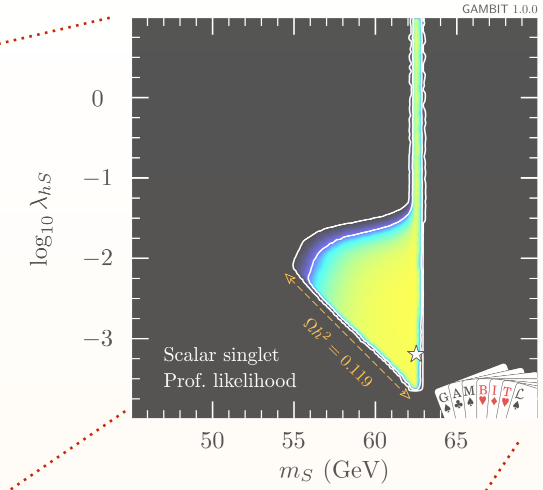
two scenarios:

QCD = A - all quarks are free and present in the plasma down to T_c =154 MeV

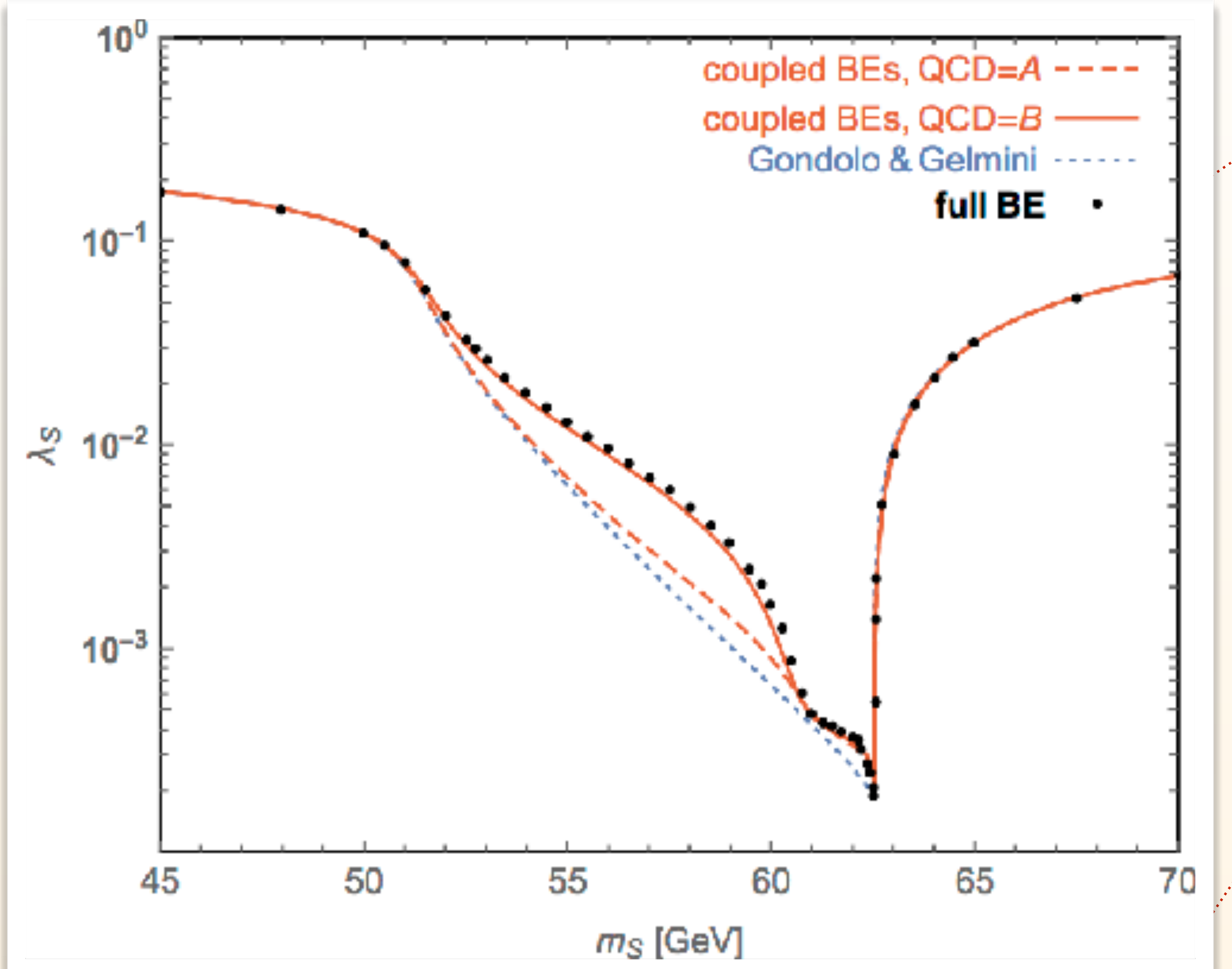
QCD = B - only light quarks contribute to scattering and only down to $4T_c$

RESULTS

RD CONTOURS



essentially the only region left for this model



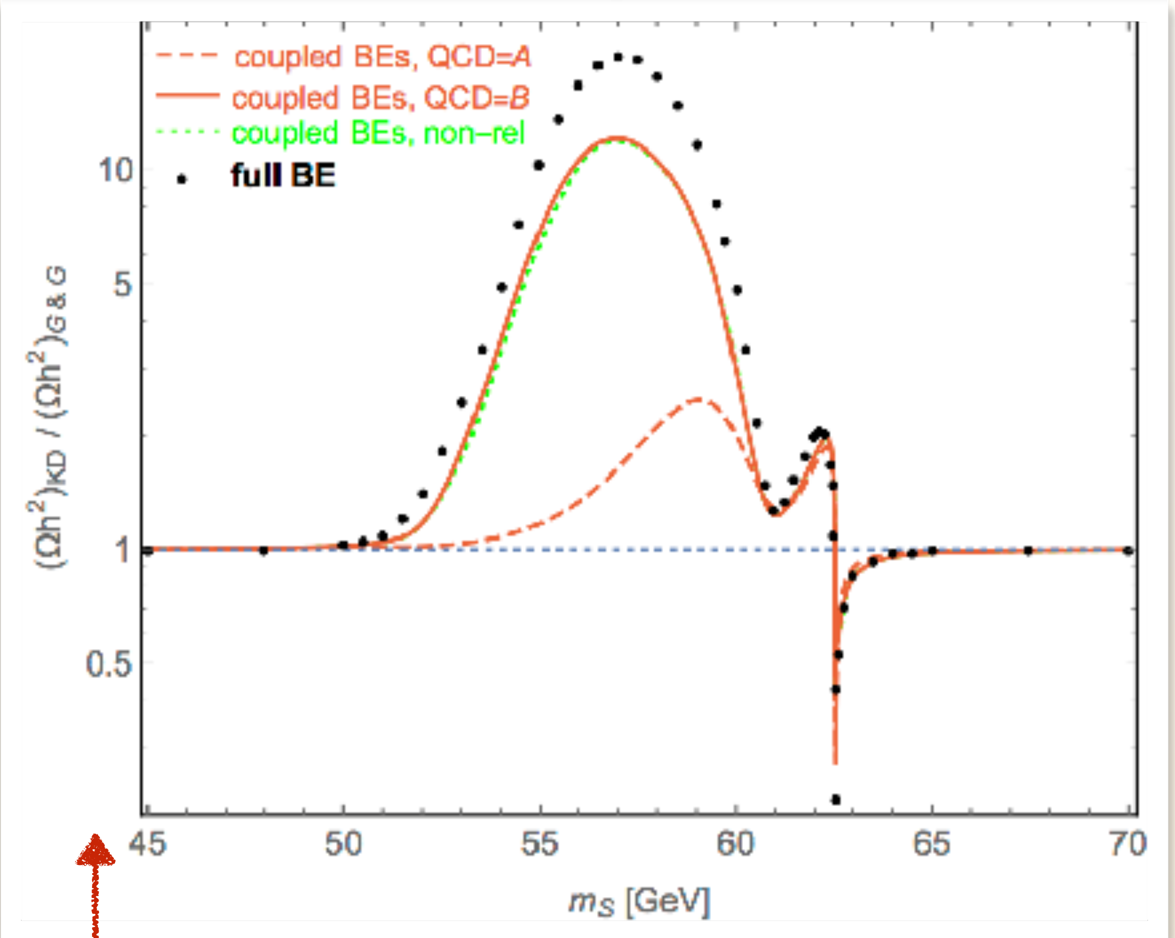
Significant modification of the observed relic density contour in the Scalar Singlet DM model

→ larger coupling needed → better chance for closing the last window

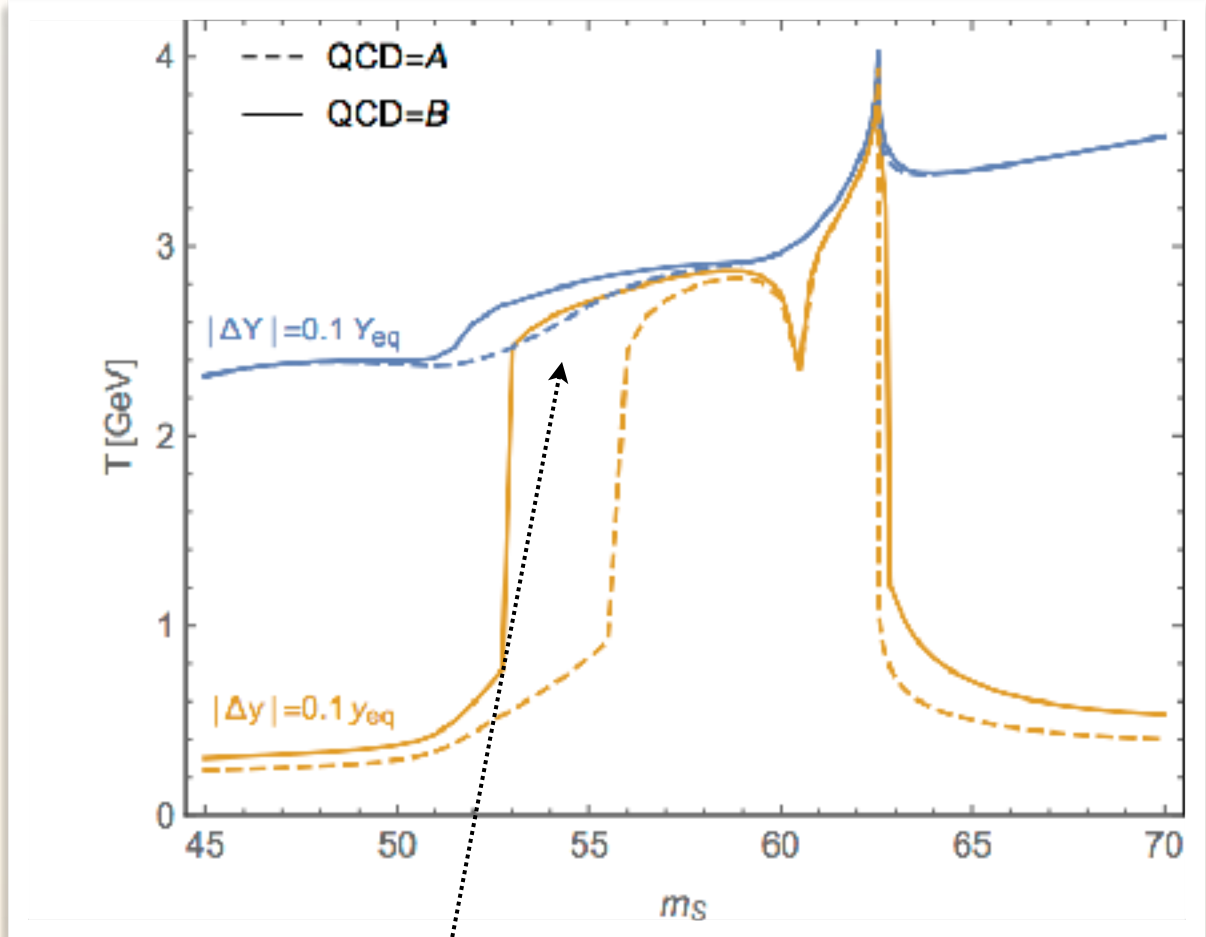
RESULTS

EFFECT

effect on relic density:



kinetic and chemical decoupling:



effect on relic density:
up to O(~10)

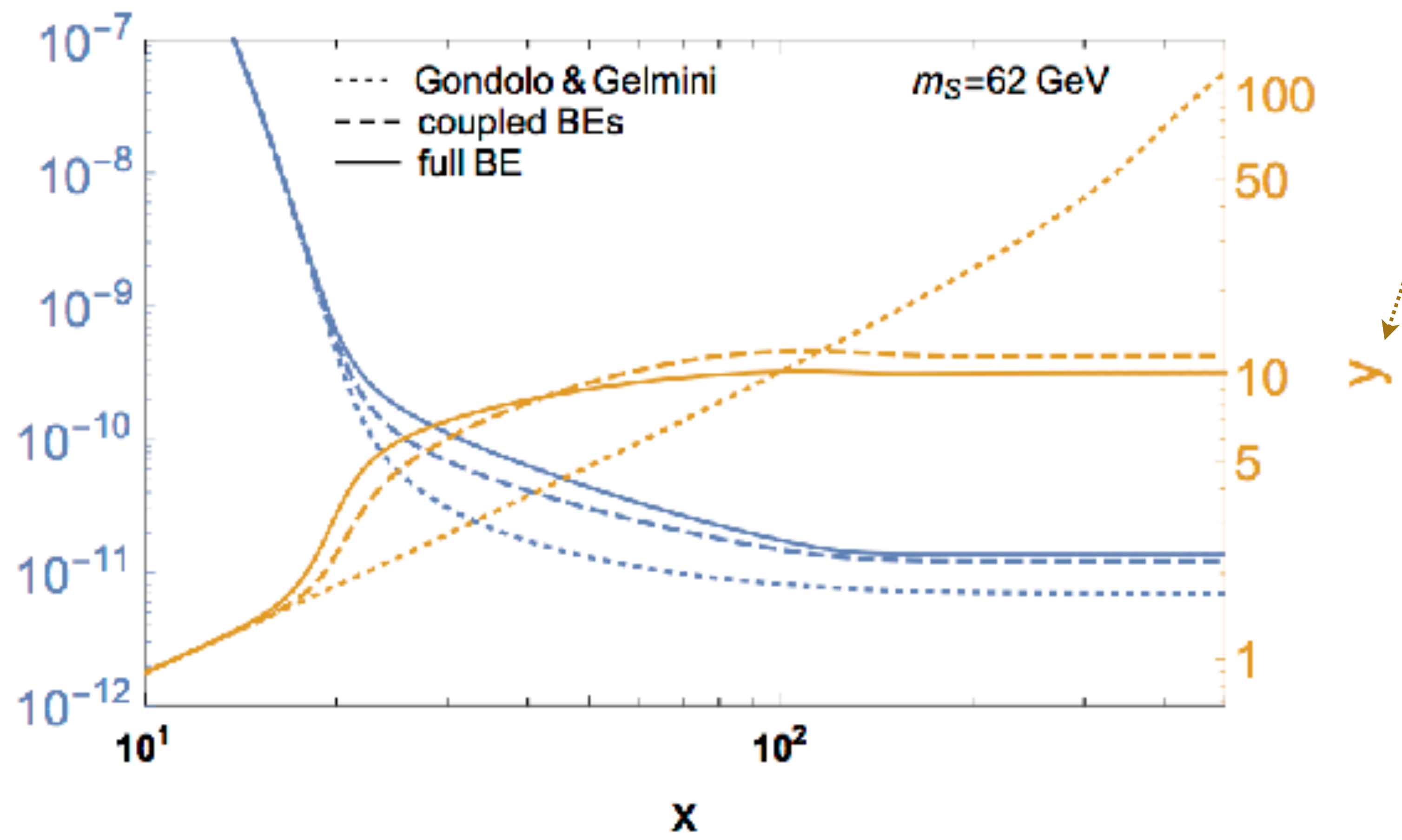
ratio approaches 1,
but does not reach it!

Why such non-trivial shape of the effect of early kinetic decoupling?

Let's inspect the y and Y evolution…

Density and T_{DM} evolution

for m_{DM} = 62 GeV, i.e. just below the resonance:



DM temperature

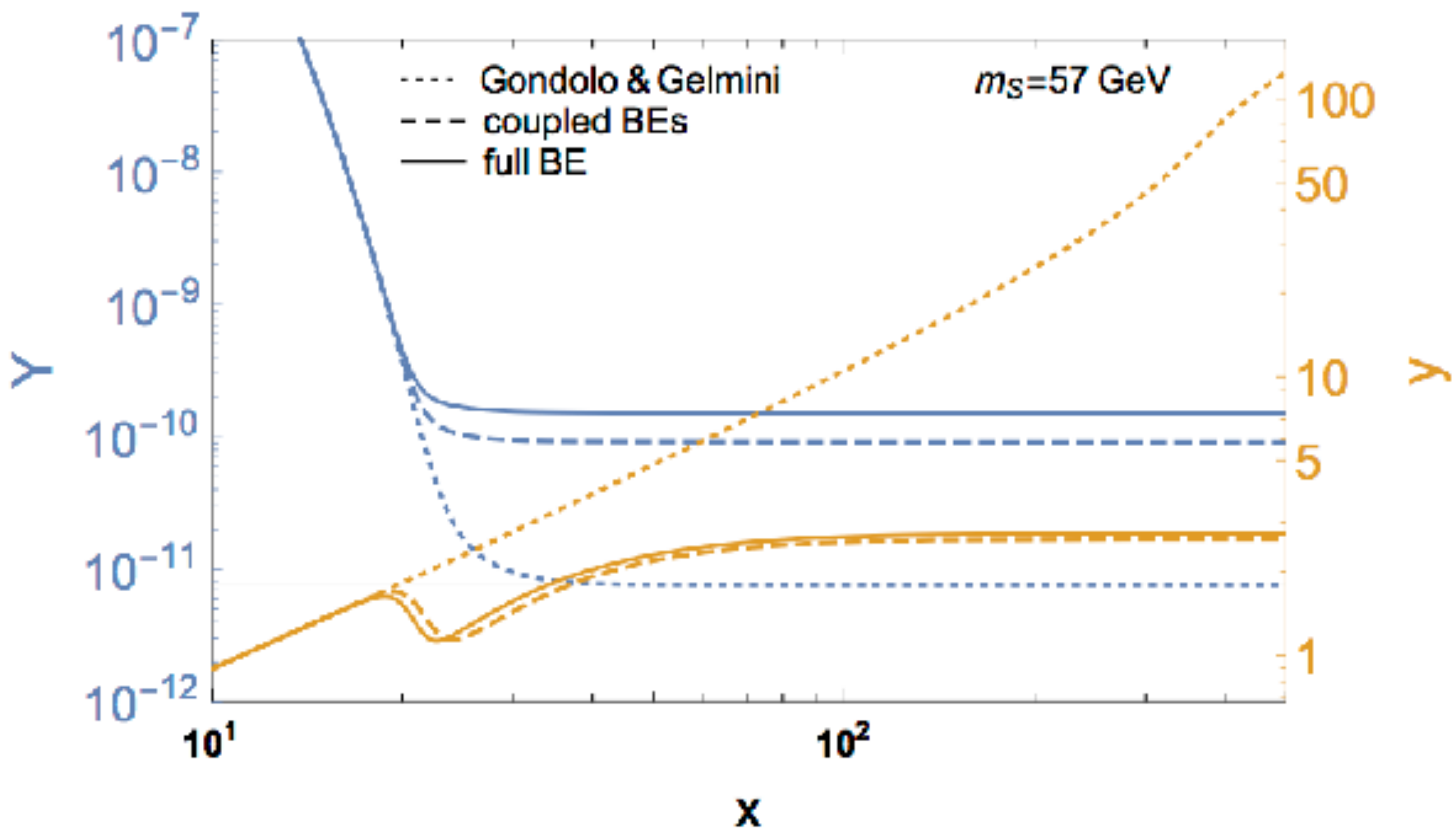
co-moving number density

Resonant annihilation **most effective** for **low momenta**

→ DM fluid goes through "heating" phase before leaves kinetic equilibrium

DENSITY AND T_{DM} EVOLUTION

for m_{DM} = 57 GeV, i.e. further away from the resonance:



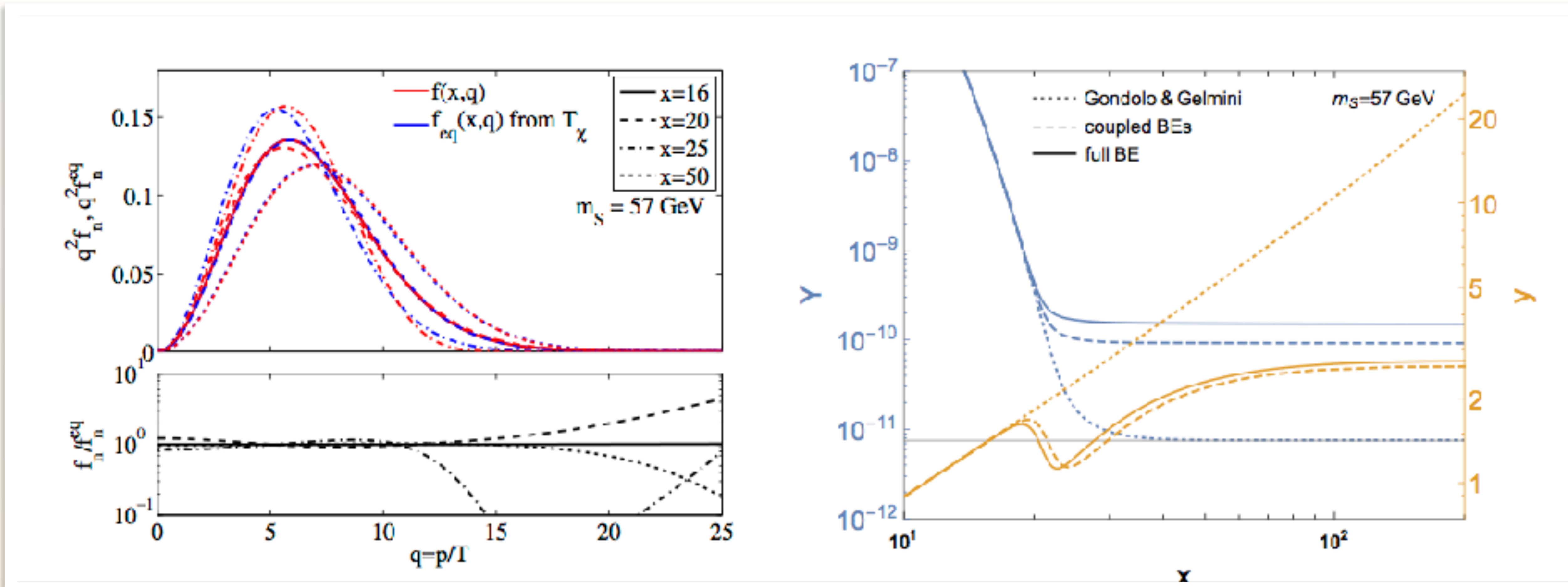
Resonant annihilation **most effective** for **high momenta**

⟶ DM fluid goes through fast "cooling" phase

after that when T_{DM} drops to much annihilation not effective anymore

FULL PHASE-SPACE BE SOLVER

Solutions for full phase-space distribution function:



Results of both approaches compatible:
some deviation from equilibrium shape mildly affects the Y and y evolution

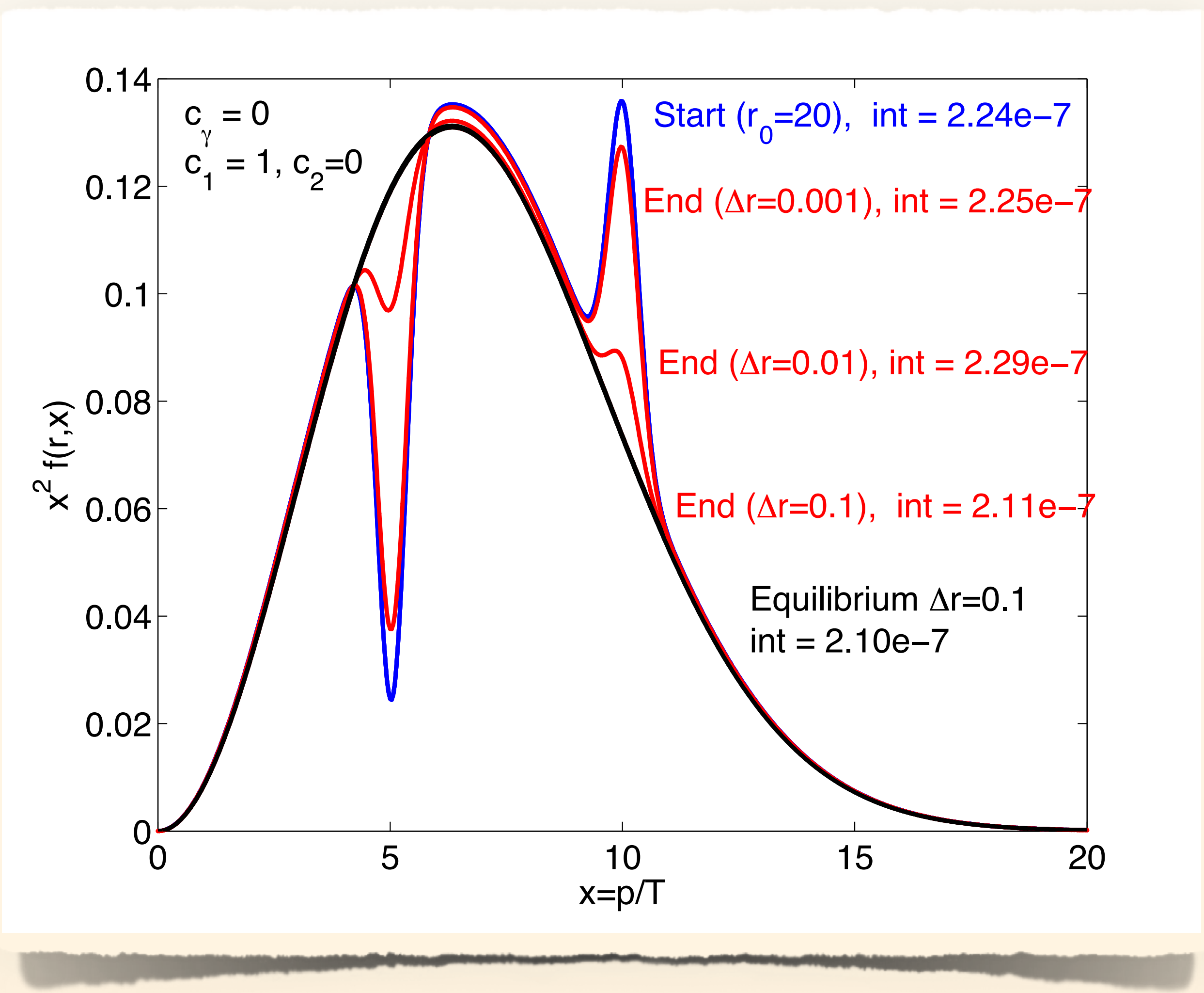
Allows to study the evolution of $f_\chi(p)$ and
the interplay between scatterings and annihilation!

KD BEFORE CD?

Obvious issue:
How to define exactly the kinetic and chemical decouplings and what is the significance of such definitions?

$\Downarrow$

Improved question:
Can kinetic decoupling happen much earlier than chemical?



we have already seen that even if scatterings were very inefficient compared to annihilation, departure from equilibrium for both Y and y happened around the same time…

turn off scatterings and take s-wave annihilation; look at local disturbance

annihilation/production precesses drive to restore kinetic equilibrium!

Summary: Part I

1. One needs to remember that kinetic equilibrium is a necessary assumption for standard relic density calculations

2. Coupled system of Boltzmann equations for 0th and 2nd moments allow for a very accurate treatment of the kinetic decoupling and its effect on relic density

3. In special cases the full phase space Boltzmann equation can be necessary — especially if one wants to trace DM temperature as well

Exception *N*:
sometimes kinetic decoupling happens together with freeze-out…

EXCEPTION n+1:

NLO EFFECTS

DARK MATTER AT NLO

Bergstrom '89; Drees et al., 9306325;
Ullio & Bergstrom, 9707333 } helicity suppression lifting

⋮

Bergstrom et al., 0507229;
Bringmann et al., 0710.3169 } spectral features in indirect searches

⋮

Ciafaloni et al., 1009.0224
Cirelli et al., 1012.4515
Ciafaloni et al., 1202.0692
AH & Iengo, 1111.2916 } large EW corrections

⋮

Chatterjee et al., 1209.2328
Harz et al., 1212.5241
Ciafaloni et al., 1305.6391
Hermann et al., 1404.2931
Boudjema et al., 1403.7459
Bringmann et al., 1510.02473
Klasen et al., 1607.06396 } ***thermal relic density***

$$\Omega_{DM} h^2 = 0.1187 \pm 0.0017.$$

<1.5% uncertainty!

Planck+WMAP pol.+highL+BAO; 1303.5062

⋮

SloopS, DM@NLO, PPC4DMID } NLO codes

Relic Density at NLO

Recall at LO:

$$C_{\mathrm{LO}} = -h_\chi^2 \int \frac{d^3\vec{p}_\chi}{(2\pi)^3} \frac{d^3\vec{p}_{\bar{\chi}}}{(2\pi)^3}\ \sigma_{\chi\bar{\chi}\to ij} v_{\mathrm{rel}}\ [f_\chi f_{\bar{\chi}}(1 \pm f_i)(1 \pm f_j) - f_i f_j (1 \pm f_\chi)(1 \pm f_{\bar{\chi}})]$$

crucial point: $$\boxed{p_\chi + p_{\bar{\chi}} = p_i + p_j \Rightarrow f_\chi^{\mathrm{eq}} f_{\bar{\chi}}^{\mathrm{eq}} \approx f_i^{\mathrm{eq}} f_j^{\mathrm{eq}}}$$

in Maxwell approx.

at NLO both virtual one-loop and 3-body processes contribute:

$$C_{1-\mathrm{loop}} = -h_\chi^2 \int \frac{d^3\vec{p}_\chi}{(2\pi)^3} \frac{d^3\vec{p}_{\bar{\chi}}}{(2\pi)^3}\ \sigma^{1-\mathrm{loop}}_{\chi\bar{\chi}\to ij} v_{\mathrm{rel}}\ [f_\chi f_{\bar{\chi}}(1 \pm f_i)(1 \pm f_j) - f_i f_j (1 \pm f_\chi)(1 \pm f_{\bar{\chi}})]$$

$$C_{\mathrm{real}} = -h_\chi^2 \int \frac{d^3\vec{p}_\chi}{(2\pi)^3} \frac{d^3\vec{p}_{\bar{\chi}}}{(2\pi)^3}\ \sigma_{\chi\bar{\chi}\to ij\gamma} v_{\mathrm{rel}}\ [f_\chi f_{\bar{\chi}}(1 \pm f_i)(1 \pm f_j)(1 + f_\gamma) - f_i f_j f_\gamma (1 \pm f_\chi)(1 \pm f_{\bar{\chi}})]$$

$$p_\chi + p_{\bar{\chi}} = p_i + p_j \pm p_\gamma \Rightarrow$$

photon can be arbitrarily soft

$$f_\gamma \sim \omega^{-1}$$

Maxwell approx. not valid anymore...

...problem: *T*-dependend IR divergence!

RELIC DENSITY

WHAT REALLY HAPPENS AT NLO?

Beneke, Dighera, AH, 1409.3049

only this used in NLO literature so far

$$C_{\rm NLO} \sim \int d\Pi_{\chi\bar{\chi}ij} \;\; f_\chi f_{\bar{\chi}} \; \boxed{\Big\{ |\mathcal{M}^{\rm LO}_{\chi\bar{\chi}\to ij}|^2 + |\mathcal{M}^{{\rm NLO}\;T=0}_{\chi\bar{\chi}\to ij}|^2 + \int d\Pi_\gamma |\mathcal{M}_{\chi\bar{\chi}\to ij\gamma}|^2 +}$$

$$|\mathcal{M}^{{\rm NLO}\;T\neq 0}_{\chi\bar{\chi}\to ij}|^2 + \int d\Pi_\gamma \Big[f_\gamma \left(|\mathcal{M}_{\chi\bar{\chi}\to ij\gamma}|^2 + |\mathcal{M}_{\chi\bar{\chi}\gamma\to ij}|^2 \right)$$

$$- f_i \left(|\mathcal{M}_{\chi\bar{\chi}\to ij\gamma}|^2 + |\mathcal{M}_{\chi\bar{\chi}i\to j\gamma}|^2 \right) - f_j \left(|\mathcal{M}_{\chi\bar{\chi}\to ij\gamma}|^2 + |\mathcal{M}_{\chi\bar{\chi}j\to i\gamma}|^2 \right) \Big] \Big\}$$

photon absorption

thermal 1-loop

photon emission

SM fermions emission

SM fermions absorption

$$- f_i f_j \Big\{ |\mathcal{M}^{\rm LO}_{ij\to\chi\bar{\chi}}|^2 + |\mathcal{M}^{{\rm NLO}\;T=0}_{ij\to\chi\bar{\chi}}|^2 + \int d\Pi_\gamma |\mathcal{M}_{ij\to\chi\bar{\chi}\gamma}|^2 +$$

$$|\mathcal{M}^{{\rm NLO}\;T\neq 0}_{ij\to\chi\bar{\chi}}|^2 + \int d\Pi_\gamma \Big[f_\gamma \left(|\mathcal{M}_{ij\to\chi\bar{\chi}\gamma}|^2 + |\mathcal{M}_{ij\gamma\to\chi\bar{\chi}}|^2 \right)$$

$$- f_\chi \left(|\mathcal{M}_{ij\to\chi\bar{\chi}\gamma}|^2 + |\mathcal{M}_{ij\chi\to\chi\gamma}|^2 \right) - f_{\bar{\chi}} \left(|\mathcal{M}_{ij\to\chi\bar{\chi}\gamma}|^2 + |\mathcal{M}_{ij\bar{\chi}\to\bar{\chi}\gamma}|^2 \right) \Big] \Big\}$$

QUESTIONS:

1. how the (soft and collinear) IR divergence cancellation happen?

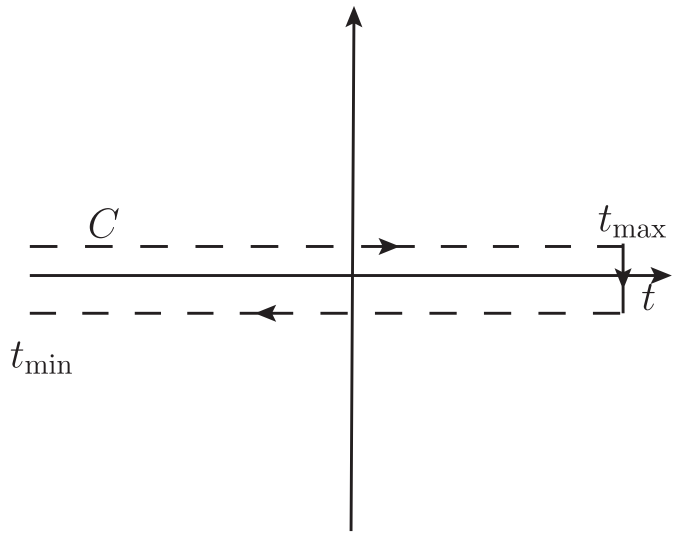
2. does Boltzmann equation itself receive quantum corrections?

3. how large are the remaining finite T corrections?

Program: develop a method for relic density calculation directly from QFT and free from IR problems

framework exists: non-equilibrium thermal field theory

Closed Time Path
formalism



$$i\Delta(x,y) = \langle T_C \phi(x)\phi^\dagger(y)\rangle,$$
$$iS_{\alpha\beta}(x,y) = \langle T_C \psi_\alpha(x)\bar{\psi}_\beta(y)\rangle,$$

contour Green's functions obey Dyson-Schwinger eqs:

$$\Delta(x,y) = \Delta_0(x,y) - \int_C d^4z \int_C d^4z' \Delta_0(x,z)\Pi(z,z')\Delta(z',y),$$
$$S_{\alpha\beta}(x,y) = S^0_{\alpha\beta}(x,y) - \int_C d^4z \int_C d^4z' S^0_{\alpha\gamma}(x,z)\Sigma_{\gamma\rho}(z,z')S_{\rho\beta}(z',y),$$

which can be rewritten in the form of Kadanoff-Baym eqs:

$$(-\partial^2 - m_\phi^2)\Delta^{\lessgtr}(x,y) - \int d^4z \Big(\Pi_h(x,z)\Delta^{\lessgtr}(z,y) - \Pi^{\lessgtr}(x,z)\Delta_h(z,y)\Big) = \mathcal{C}_\phi,$$
$$(i\partial\!\!\!/ - m_\chi)S^{\lessgtr}(x,y) - \int d^4z \Big(\Sigma_h(x,z)S^{\lessgtr}(z,y) - \Sigma^{\lessgtr}(x,z)S_h(z,y)\Big) = \mathcal{C}_\chi$$

Closed Time Path

path to Boltzmann equation

Kadanoff-Baym $\Longrightarrow$ Boltzmann

$$E\left(\partial_t - H\vec{p}\cdot\nabla_{\vec{p}}\right) f = \mathcal{C}[f].$$

collision term **derived** from thermal QFT

Assumptions:

weak inhomogeneity

gradient expansion

quasi-particle approx.

weak interactions

Justification:

inhomogeneity

plasma excitation momenta

$$\partial \ll k$$

freeze-out happens close to equilibrium

Closed Time Path

FORMALISM: COLLISION TERM

the fermion collision terms is defined as:

propagators

$$\mathcal{C}_\chi = \frac{1}{2}\int d^4z\,\left(\boxed{\Sigma^>(x,z)}\boxed{S^<(z,y)} - \boxed{\Sigma^<(x,z)}\boxed{S^>(z,y)}\right)$$

self-energies

where the propagators:

thermal part

$$iS^c(p) = \frac{i\left(\not p + m\right)}{p^2 - m^2 + i\eta} - \boxed{2\pi\left(\not p + m\right)\delta\left(p^2 - m^2\right) f\left(p^0\right)}$$

$$iS^a(p) = -\frac{i\left(\not p + m\right)}{p^2 - m^2 + i\eta} + 2\pi\left(\not p + m\right)\delta\left(p^2 - m^2\right)\left(1 - f\left(p^0\right)\right)$$

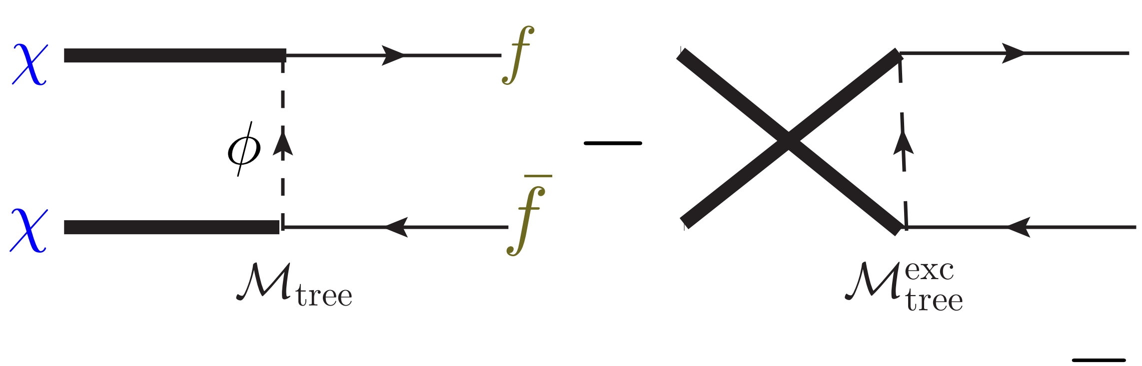
$$\left.\begin{aligned} iS^>(p) &= 2\pi\left(\not p + m\right)\delta\left(p^2 - m^2\right)\left(1 - f\left(p^0\right)\right) \\ iS^<(p) &= -2\pi\left(\not p + m\right)\delta\left(p^2 - m^2\right) f\left(p^0\right) \end{aligned}\right\}\ \text{"cut" propagators}$$

the presence of distribution functions inside propagators $\Rightarrow$ known collision term structure

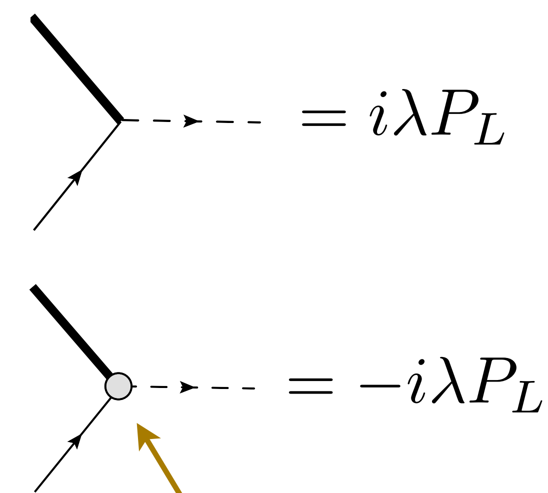
Collision Term
example

Bino-like DM: χ Majorana fermion, SM singlet

annihilation process at tree level:



vertices (2 types):



dotted

scale hierarchy: $\boxed{m_\phi \gtrsim m_\chi} \gg T \gg \boxed{m_f}$

no thermal contributions

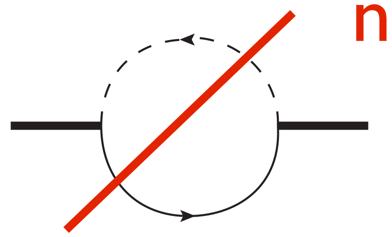
effectively massless

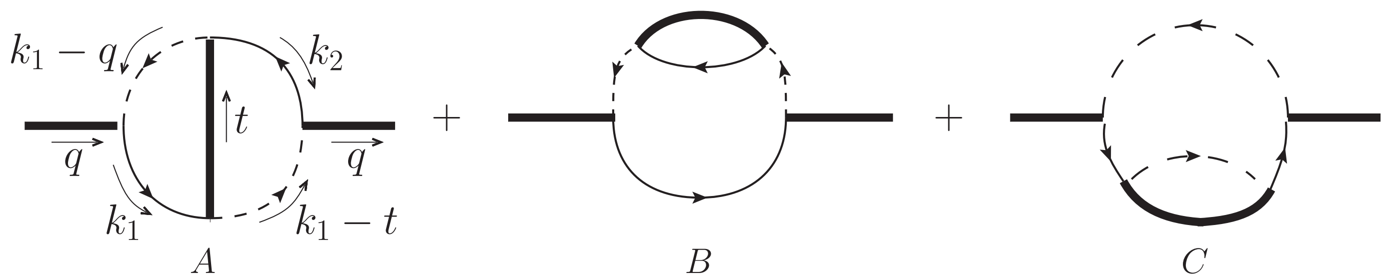
rescaled variables: $\tau = \dfrac{T}{m_\chi} \ll 1$ $\quad\epsilon = \dfrac{m_f}{2m_\chi} \ll \tau$ $\quad\xi = \dfrac{m_\phi}{m_\chi} \gtrsim 1$

Collision Term
computation

no # changing processes

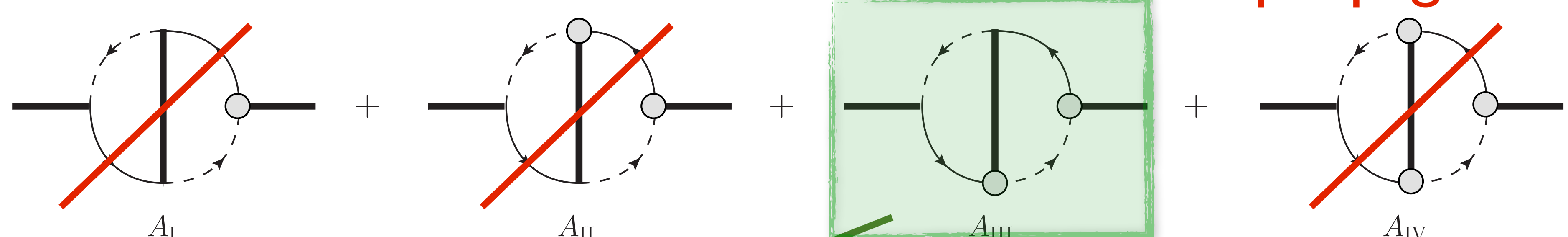
summed over dotted and undotted indices

$i\Sigma_1 =$ 

$i\Sigma_2 =$ 

$\Downarrow$

cut scalar propagator

$i\Sigma_A^> =$ 

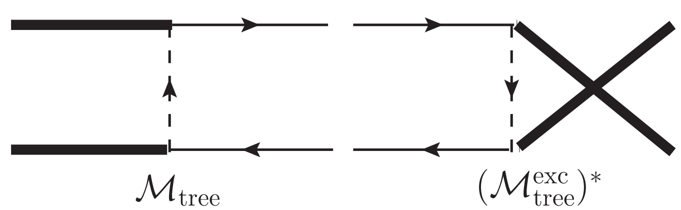
$$\Sigma^>_{A_{\mathrm{III}}}(q)\,S^<(q) = -\lambda^4 \int \frac{d^4t}{(2\pi)^4}\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}(2\pi)^4\,\delta(q+t-k_1-k_2)$$
$$\underbrace{i\Delta^{11}(k_1-q)\,i\Delta^{22}(k_1-t)}_{\equiv\mathcal{S}}\underbrace{P_{\mathrm{R}}iS^{21}(k_2)\,P_{\mathrm{L}}iS^{12}(t)\,P_{\mathrm{L}}iS^{21}(k_1)\,P_{\mathrm{R}}iS^{12}(q)}_{\equiv\mathcal{F}}$$

Collision Term
matching

after inserting the propagators:

$$\Sigma^{>}_{A_{\mathrm{III}}}(q)\,S^{<}(q) = \frac{1}{2E_{\chi_1}}(2\pi)\,\delta\left(q^0 - E_{\chi_1}\right)\int\frac{d^4t}{(2\pi)^3\,2E_{\chi_2}}\delta\left(t^0 - E_{\chi_2}\right)\times$$
$$\int\frac{d^3\vec{k}_1}{(2\pi)^3\,2E_{f_1}}\frac{d^3\vec{k}_2}{(2\pi)^3\,2E_{f_2}}(2\pi)^4\,\delta\left(q+t-k_1-k_2\right)|\mathcal{M}_A|^2\left[f_\chi(q)\,f_\chi(t)\left(1-f_f^{\mathrm{eq}}\left(k_1^0\right)\right)\left(1-f_f^{\mathrm{eq}}\left(k_2^0\right)\right)\right]$$

$\Rightarrow$ one indeed recovers the known collision term and

$|\mathcal{M}_A|^2 =$ 

(part of) tree level $|\mathcal{M}|^2$

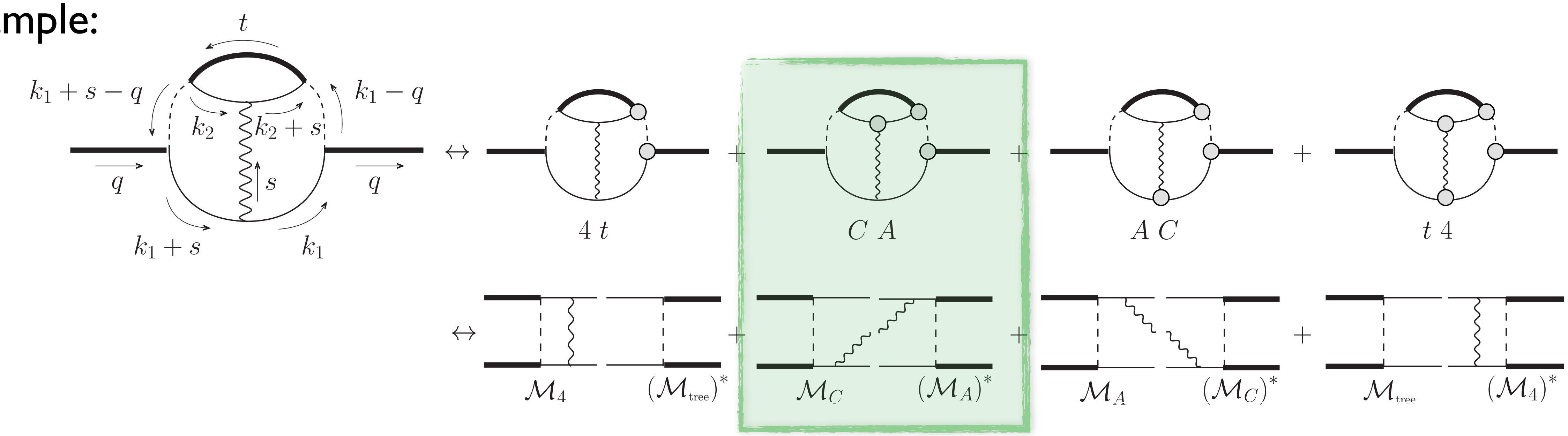
repeating the same for B type diagrams the bottom line:

$i\Sigma^{>} \leftrightarrow$ tree level annihilation contribution to the collision term

Collision Term
matching at NLO

$i\Sigma_3 =$ 20 self-energy diagrams

example:



$$\Sigma^{>}_{\text{CA}}(q)\, S^{<}(q) = \frac{1}{2E_{\chi_1}}(2\pi)\,\delta\left(q^0 - E_{\chi_1}\right)\int \frac{d^4 t}{(2\pi)^3\, 2E_{\chi_2}}\delta\left(t^0 - E_{\chi_2}\right)$$

$$\int \frac{d^3\vec{k}_1}{(2\pi)^3\, 2E_{f_1}}\frac{d^3\vec{k}_2}{(2\pi)^3\, 2E_{f_2}}\frac{d^3\vec{s}}{(2\pi)^3\, 2E_{\gamma}}(2\pi)^4\,\delta\left(q+t-k_1-k_2-s\right)$$

$$\mathcal{M}_C\left(\mathcal{M}_A\right)^*\left[f_\chi(q)\, f_\chi(t)\left(1 - f_f^{\text{eq}}\left(k_1^0\right)\right)\left(1 - f_f^{\text{eq}}\left(k_2^0\right)\right)\left(1 + f_\gamma^{\text{eq}}\left(s^0\right)\right)\right]$$

$\Rightarrow$ at NLO thermal effects do **not** change the collision therm structure

Results

every contribution can be written in a form:

$$\int_0^\infty d\omega\, f_\gamma(\omega) \boxed{S(\omega, e_\chi, \epsilon, \xi)} \qquad f_\gamma(\omega) = \frac{1}{1-e^{\omega/T}}$$

photon energy

expand in ω

$$S = \sum_{i=-1}^{\infty} s_n \omega^n$$

note:

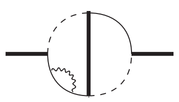
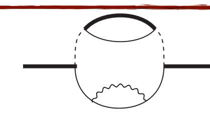
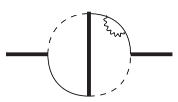
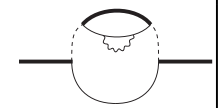
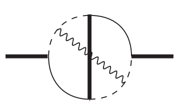
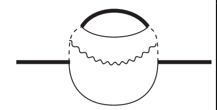
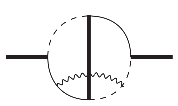
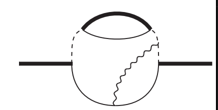
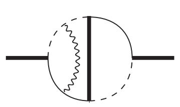
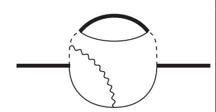
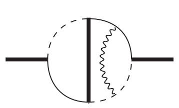
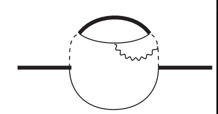
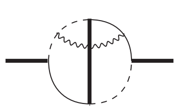
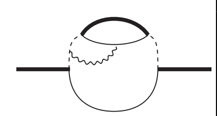
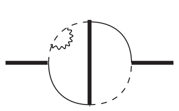
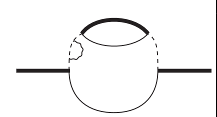
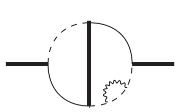
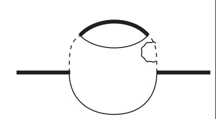
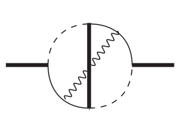
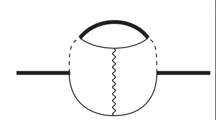
$$J_n \equiv \int_0^\infty f_B(\omega)\omega^n d\omega = \begin{cases} \text{div} & n \leq 0 \\ \sim \tau^{n+1} & n > 0 \end{cases}$$

IR divergence in separate terms: $J_{-1} \leftrightarrow T = 0$ soft div
$J_0 \leftrightarrow T = 0$ soft eikonal

finite T corrections: $J_1 \leftrightarrow \mathcal{O}(\tau^2)$ ...

Results

IR divergence cancellation: s-wave

The divergent part J_{-1}							
Type A	Real	Virtual	External	Type B	Real	Virtual	External
	$\frac{\alpha(1-2\epsilon^2)}{\pi\epsilon^2}$		$-\frac{\alpha(1-2\epsilon^2)}{\pi\epsilon^2}$		$-\frac{\alpha}{\pi\epsilon^2}$		$\frac{\alpha}{\pi\epsilon^2}$
	$\frac{\alpha(1-2\epsilon^2)}{\pi\epsilon^2}$		$-\frac{\alpha(1-2\epsilon^2)}{\pi\epsilon^2}$		$-\frac{\alpha}{\pi\epsilon^2}$		$\frac{\alpha}{\pi\epsilon^2}$
	0				0		
	0	0			0	0	
	0	0			0	0	
	0	0			0	0	
	0	0			0	0	
		0				0	
		0				0	
	$\frac{2\alpha(1-2\epsilon^2)^2}{\pi\epsilon^2\sqrt{1-4\epsilon^2}}L$	$-\frac{2\alpha(1-2\epsilon^2)^2}{\pi\epsilon^2\sqrt{1-4\epsilon^2}}L$			$\frac{2\alpha(1-2\epsilon^2)}{\pi\epsilon^2\sqrt{1-4\epsilon^2}}L$	$-\frac{2\alpha(1-2\epsilon^2)}{\pi\epsilon^2\sqrt{1-4\epsilon^2}}L$	

$\longrightarrow$ cancels in **every** row separately

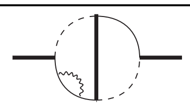
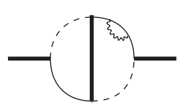
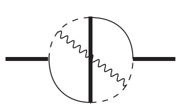
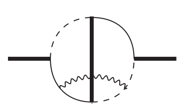
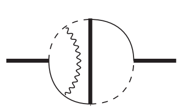
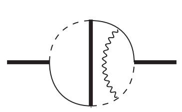
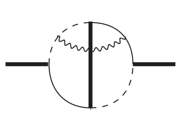
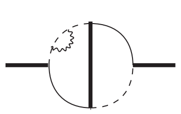
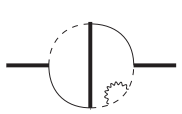
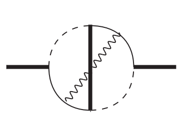
$\Longrightarrow$ every CTP self-energy is IR finite

Results

finite T correction: s-wave

factorized $\frac{\pi}{6}\alpha\tau^2\frac{a_{\rm tree}}{\epsilon^2}$

The finite part J_1

Type A	Real	Virtual	External
	$\frac{2(1-\xi^2)}{D^2D_\xi^2}+\frac{(1-2\epsilon^2)p_1(\epsilon,\xi)}{2D^2D_\xi^2}\boxed{+\frac{1}{2\sqrt{D}}L}$		$\frac{(1-2\epsilon^2)(\xi^2-3D)}{2DD_\xi}\boxed{-\frac{1}{2\sqrt{D}}L}$
	— " —		— " —
	$-\frac{4(1-2\epsilon^2)D}{D_\xi^2}$		
	$-\frac{2(1-2\epsilon^2)\xi^2}{D_\xi^2}-\frac{f_1(\epsilon,\xi)}{\sqrt{D}D_\xi^2}L$	$\frac{2(1-2\epsilon^2)(D-\xi^2)}{D_\xi^2}+\frac{f_1(\epsilon,\xi)}{\sqrt{D}D_\xi^2}L$	
	— " —	— " —	
	— " —	— " —	
	— " —	— " —	
		$-\frac{4(1-2\epsilon^2)D}{D_\xi^2}$	
		— " —	
	$\frac{2(1-2\epsilon^2)p_2(\epsilon,\xi)+(1-\xi^2)^2}{D^2D_\xi^2}+\frac{4f_2(\epsilon,\xi)}{\sqrt{D}D_\xi^2}L$	$\frac{16\epsilon^2(2-3\epsilon^2)-(3-\xi^2)^2}{D_\xi^2}-\frac{4f_2(\epsilon,\xi)}{\sqrt{D}D_\xi^2}L$	

Log terms cancels in **every** row separately

$\Downarrow$

no collinear divergence!

$\xi=\frac{m_\phi}{m_\chi}\gtrsim 1$

$\tau=\frac{T}{m_\chi}\ll 1$

$\epsilon=\frac{m_f}{2m_\chi}\ll \tau$

separate contributions complicated, but when summed up...

$$\Delta a_{\tau^4}^{\epsilon=0}=\frac{8\pi^2\lambda^4\alpha\tau^4}{45}\frac{1}{(1+\xi^2)^4}=\frac{4\pi}{45}\alpha\tau^4\frac{1}{(1+\xi^2)^2}\frac{a_{\rm tree}}{\epsilon^2}\Big|_{\epsilon=0}$$

strongly suppressed as at kinetic equilibrium $\tau\sim v^2$

The power of Thermal OPE

M. Beneke, F. Dighera, AH, 1607.03910

The cross section can be written as the Im part of the forward scattering amplitude:

$$\sigma v_{\rm rel} = \frac{2}{s}\,{\rm Im}\left\{(-i)\boxed{\int d^4x\, \frac{1}{4}\sum_{\rm spin}\langle\bar\chi\chi;T|\,\mathcal{T}\left\{\mathcal{O}_{\rm ann}(0)\,\mathcal{O}^\dagger_{\rm ann}(x)\right\}\,|\bar\chi\chi;T\rangle}\right\}$$

clear separation of soft (thermal effects) and hard (annihilation/decay) modes

$$T \ll m$$

$\Rightarrow$ **Operator Product Expansion**

$$-i\int d^4x\; e^{-ip\cdot x}\;\mathcal{T}\left\{J^\mu_A(0)\,J^{\nu\dagger}_B(x)\right\} = \sum_i C^i_{AB}(p)\cdot\mathcal{O}_i$$

Wilson coeffs. matched at T=0

Possible operators up to dim 4:

$$\mathbb{1}\;,\qquad F^{\alpha\beta}F^{\gamma\delta}\;,\qquad m_f\bar f\,\Gamma f\;,\qquad \bar f\,\Gamma\, iD^\alpha f$$

Matrix elements: LO, $\mathcal{O}(\alpha T^4)$, $\mathcal{O}(\alpha m_f^2 T^2)$, $\mathcal{O}(\alpha T^4)$

No dim 2 operator! No IR divergence to begin with!

ADVANTAGES OF OPE

- The scaling with T is manifest
- Separation of T=0 and T-dependent contributions
- Significant simplification of the computations
- Clear physics interpretation: at $\mathcal{O}(\alpha\tau^2)$ effects of thermal kinetic energy

Example: muon decay in thermal bath*

Czarnecki et al. '11

$$-i\int d^4x\, e^{-ip\cdot x}\, \mathcal{T}\{J^\mu(0)\, J^{\nu\dagger}(x)\} = C_0^{\mu\nu}\bar{\psi}\psi + C_2^{\mu\nu}\cancel{\bar{\psi}\frac{i}{2}\sigma_{\alpha\beta}F^{\alpha\beta}\psi} + \mathcal{O}(m_\psi^{-3}),$$

$$\bar{\psi}\psi = \underbrace{\bar{\psi}\not{v}\psi}_{\text{LO}} + \underbrace{\boxed{\frac{1}{2m_\psi^2}\bar{\psi}\,(iD_\perp)^2\,\psi}}_{\mathcal{O}(\alpha\tau^2)} + \cancel{\frac{i}{4m_\psi^2}\bar{\psi}\sigma_{\alpha\beta}F^{\alpha\beta}\psi} + \mathcal{O}(m_\psi^{-3}),$$

…and the final correction:

$$\Gamma_T = \Gamma_0\,(1 - \boxed{K_\psi}) + \mathcal{O}(T^3/m_\psi^3).$$

*Analogy: semi-leptonic H_b decay in QCD

In the Literature:

OPE in finite temperature - Hatsuda, Koike, Lee '93; Mallik '97; …

Related EFT approach - Biondini, Brambilla, Escobedo, Vairo '13; …

SUMMARY: PART II

1. how the (soft and collinear) IR divergence cancellation happen?
   automatic in thermal QFT formalism, cancellation at the level of every CTP self-energy

2. does Boltzmann equation itself receive quantum corrections?
   no, not at NLO

3. how large are the remaining finite T corrections?
   strongly suppressed, of order $\mathcal{O}(\alpha T^4)$

Exception *N+1*:
LO sometimes is not enough
(and then in principle $T \neq 0$ QFT needed)

...but in practice one can safely use BE with NLO cross-section

Takeaway message

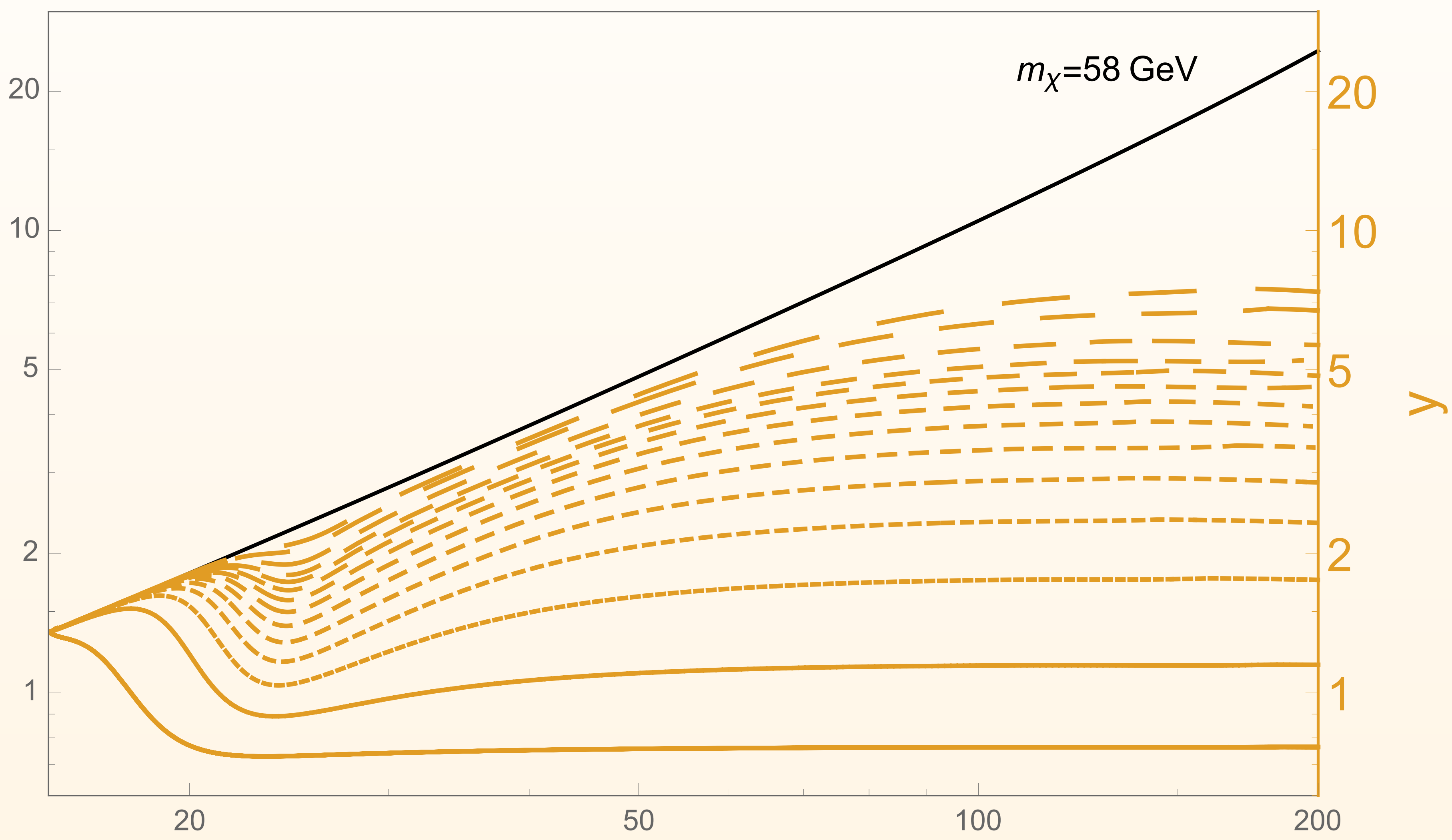
When computing relic density of dark matter one needs carefully to check if the standard treatment is sufficient for the case at hand

”Everything should be made as simple as possible, but no simpler.”

attributed to* **Albert Einstein**

*The published quote reads:
”It can scarcely be denied that the supreme goal of all theory is to make the irreducible basic elements as simple and as few as possible without having to surrender the adequate representation of a single datum of experience.”
„On the Method of Theoretical Physics" ,The Herbert Spencer Lecture, delivered at Oxford (10 June 1933); also published in *Philosophy of Science*, Vol. 1, No. 2 (April 1934), pp. 163-169., p. 165

BACKUP



m_χ=58 GeV
20
10
5
2
1
20
50
100
200
20
10
5
2
1
y