Relic Density at NLO:
the thermal IR divergence

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based on: M. Beneke, F. Dighera, A.H., 1407.????

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DARK MATTER AT NLO

Bergstrom '89; Drees et al., 9306325; Ullio & Bergstrom, 9707333	helicity suppression lifting
⋮	
Bergstrom et al., 0507229; Bringmann et al., 0710.3169	spectral features in indirect searches
⋮	
Ciafaloni et al., 1009.0224 Cirelli et al., 1012.4515 Ciafaloni et al., 1202.0692 A.H. & Iengo, 1111.2916	large EW corrections
⋮	
Chatterjee et al., 1209.2328 Harz et al., 1212.5241 Ciafaloni et al., 1305.6391 Hermann et al., 1404.2931 Boudjema et al., 1403.7459	***thermal relic density***
⋮	
SloopS, DM@NLO, PPC4DMID	NLO codes

$$\Omega_{DM} h^2 = 0.1187 \pm 0.0017.$$

<1.5% uncertainty!

Planck+WMAP pol.+highL+BAO; 1303.5062

Relic Density:
standard approach

T ↑

$\Gamma_{\rm ann} > H$ — DM in equilibrium

$\Gamma_{\rm ann} \sim H$ — chemical decoupling

$\Gamma_{\rm ann} < H$ — freeze-out

↓ time

non-equilibrium

time evolution of $f_\chi(p)$ in kinetic theory:

$$E\left(\partial_t - H\vec{p}\cdot\nabla_{\vec{p}}\right) f_\chi = \mathcal{C}[f_\chi] \quad \Rightarrow \quad \frac{dn_\chi}{dt} + 3Hn_\chi = C$$

Liouville operator in FRW background

the collision term

integrated

for $2 \leftrightarrow 2$ CP invariant process:

$$C_{\rm LO} = -h_\chi^2 \int \frac{d^3\vec{p}_\chi}{(2\pi)^3} \frac{d^3\vec{p}_{\bar\chi}}{(2\pi)^3}\, \sigma_{\chi\bar\chi\to ij} v_{\rm rel}\, \left[f_\chi f_{\bar\chi}(1\pm f_i)(1\pm f_j) - f_i f_j (1\pm f_\chi)(1\pm f_{\bar\chi})\right]$$

note: added "by hand"

Relic Density:
what happens at NLO?

at NLO both virtual one-loop and 3-body processes contribute:

$$C_{1-\mathrm{loop}} = -h_\chi^2 \int \frac{d^3\vec{p}_\chi}{(2\pi)^3}\frac{d^3\vec{p}_{\bar\chi}}{(2\pi)^3}\; \sigma^{1-\mathrm{loop}}_{\chi\bar\chi\to ij} v_{\mathrm{rel}}\; [f_\chi f_{\bar\chi}(1\pm f_i)(1\pm f_j) - f_i f_j (1\pm f_\chi)(1\pm f_{\bar\chi})]$$

$$C_{\mathrm{real}} = -h_\chi^2 \int \frac{d^3\vec{p}_\chi}{(2\pi)^3}\frac{d^3\vec{p}_{\bar\chi}}{(2\pi)^3}\; \sigma_{\chi\bar\chi\to ij\gamma} v_{\mathrm{rel}}\; [f_\chi f_{\bar\chi}(1\pm f_i)(1\pm f_j)(1+f_\gamma) - f_i f_j f_\gamma (1\pm f_\chi)(1\pm f_{\bar\chi})]$$

$$p_\chi + p_{\bar\chi} = p_i + p_j \pm p_\gamma \Rightarrow$$

photon can be arbitrarily soft

$f_\gamma \sim \omega^{-1}$

Maxwell approx. not valid...

...problem: IR divergence

Relic Density:
what happens at NLO?

the correct expression at NLO including QED corrections:

only this used in NLO literature so far

$$C_{\rm NLO} \sim \int d\Pi_{\chi\bar{\chi}ij} \; f_\chi f_{\bar{\chi}} \; \Big\{ |\mathcal{M}^{\rm LO}_{\chi\bar{\chi}\to ij}|^2 + |\mathcal{M}^{{\rm NLO}\ T=0}_{\chi\bar{\chi}\to ij}|^2 + \int d\Pi_\gamma |\mathcal{M}_{\chi\bar{\chi}\to ij\gamma}|^2 +$$

$$|\mathcal{M}^{{\rm NLO}\ T\neq 0}_{\chi\bar{\chi}\to ij}|^2 + \int d\Pi_\gamma \left[f_\gamma \left(|\mathcal{M}_{\chi\bar{\chi}\to ij\gamma}|^2 + |\mathcal{M}_{\chi\bar{\chi}\gamma\to ij}|^2 \right) \right.$$

$$-f_i f_j \Big\{ |\mathcal{M}^{\rm LO}_{ij\to\chi\bar{\chi}}|^2 + |\mathcal{M}^{{\rm NLO}\ T=0}_{ij\to\chi\bar{\chi}}|^2 + \int d\Pi_\gamma |\mathcal{M}_{ij\to\chi\bar{\chi}\gamma}|^2 +$$

$$|\mathcal{M}^{{\rm NLO}\ T\neq 0}_{ij\to\chi\bar{\chi}}|^2 + \int d\Pi_\gamma \left[f_\gamma \left(|\mathcal{M}_{ij\to\chi\bar{\chi}\gamma}|^2 + |\mathcal{M}_{ij\gamma\to\chi\bar{\chi}}|^2 \right) \right.$$

thermal 1-loop

photon emission

photon absorption

QUESTIONS

1. how the (soft and collinear) IR divergence cancellation happen?

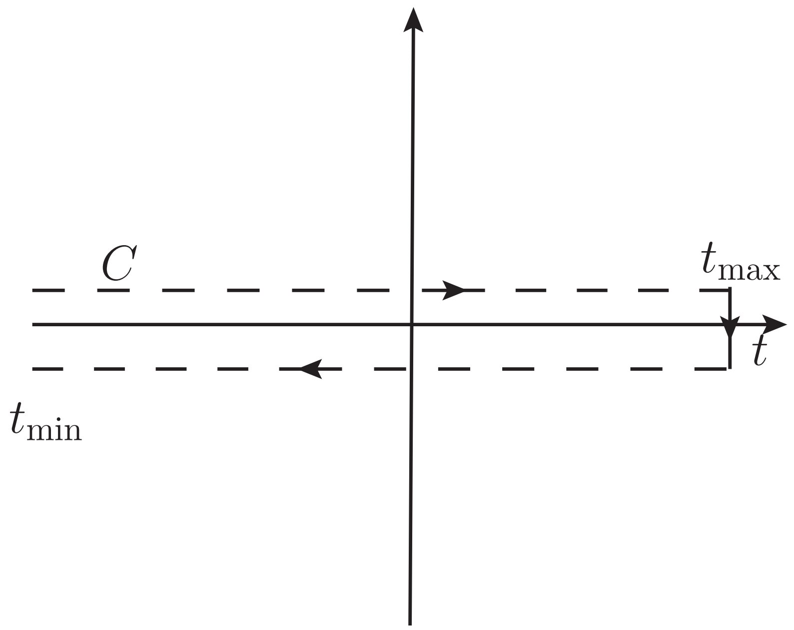
2. does Boltzmann equation itself receive quantum corrections?

3. how large are the remaining finite T corrections?

Program: develop a method for relic density calculation directly from QFT and free from IR problems

framework exists: non-equilibrium thermal field theory

CLOSED TIME PATH
FORMALISM



Def. contour fermion Green's function:

$$iS_{\alpha\beta}(x,y) = \langle T_C \psi_\alpha(x) \bar{\psi}_\beta(y) \rangle,$$

contour Green's functions obey Dyson-Schwinger eq, whose kinetic part can be rewritten in the form of Kadanoff-Baym eqs:

$$(i\partial\!\!\!/ - m_\chi) S^{\lessgtr}(x,y) - \int d^4z \left(\Sigma_h(x,z) S^{\lessgtr}(z,y) - \Sigma^{\lessgtr}(x,z) S_h(z,y) \right) = \mathcal{C}_\chi,$$

Closed Time Path

path to Boltzmann equation

Kadanoff-Baym $\Longrightarrow$ Boltzmann

$$E\left(\partial_t - H\vec{p}\cdot\nabla_{\vec{p}}\right) f = \mathcal{C}[f].$$

collision term **derived** from thermal QFT

Assumptions:

weak inhomogeneity → gradient expansion

weak inhomogeneity → quasi-particle approx.

weak interactions → quasi-particle approx.

Justification:

inhomogeneity

plasma excitation momenta

$$\partial \ll k$$

freeze-out happens close to equilibrium

Closed Time Path

FORMALISM: COLLISION TERM

the fermion collision terms is defined as:

propagators

$$\mathcal{C}_\chi = \frac{1}{2}\int d^4 z \left(\Sigma^>(x,z) S^<(z,y) - \Sigma^<(x,z) S^>(z,y)\right)$$

self-energies

where the propagators:

thermal "cut" part

$$iS^c(p) = \frac{i(\not{p}+m)}{p^2-m^2+i\eta} - 2\pi(\not{p}+m)\,\delta(p^2-m^2)\, f(p^0)$$

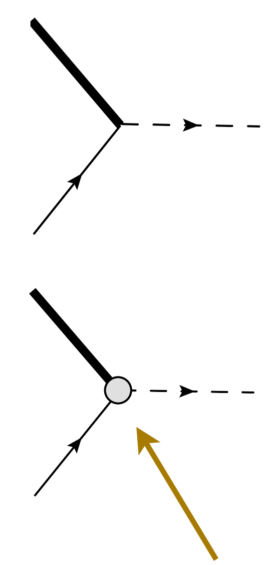
$$iS^a(p) = -\frac{i(\not{p}+m)}{p^2-m^2+i\eta} + 2\pi(\not{p}+m)\,\delta(p^2-m^2)\,(1-f(p^0))$$

$$iS^>(p) = 2\pi(\not{p}+m)\,\delta(p^2-m^2)\,(1-f(p^0))$$

$$iS^<(p) = -2\pi(\not{p}+m)\,\delta(p^2-m^2)\, f(p^0)$$

} "cut" propagators

vertices (2 types):



$= i\lambda P_L$

$= -i\lambda P_L$

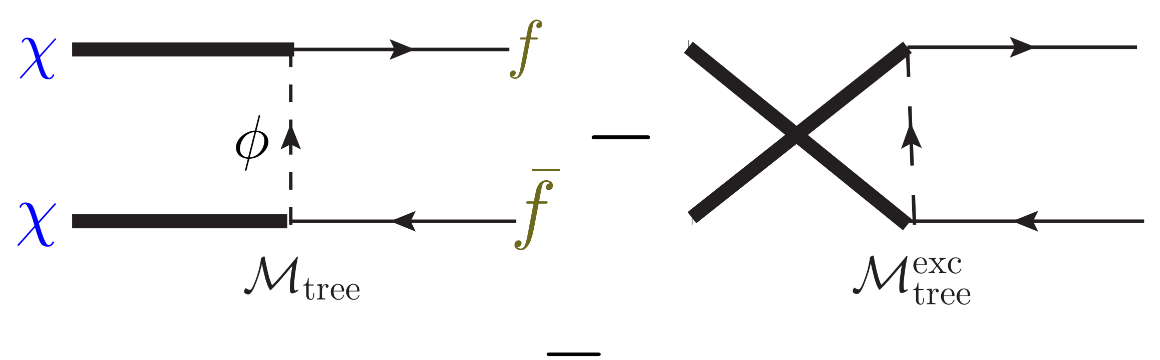
dotted

the presence of distribution functions inside propagators $\Rightarrow$ known collision term structure

Collision Term
example

Bino-like DM: χ Majorana fermion, SM singlet

annihilation process at tree level:



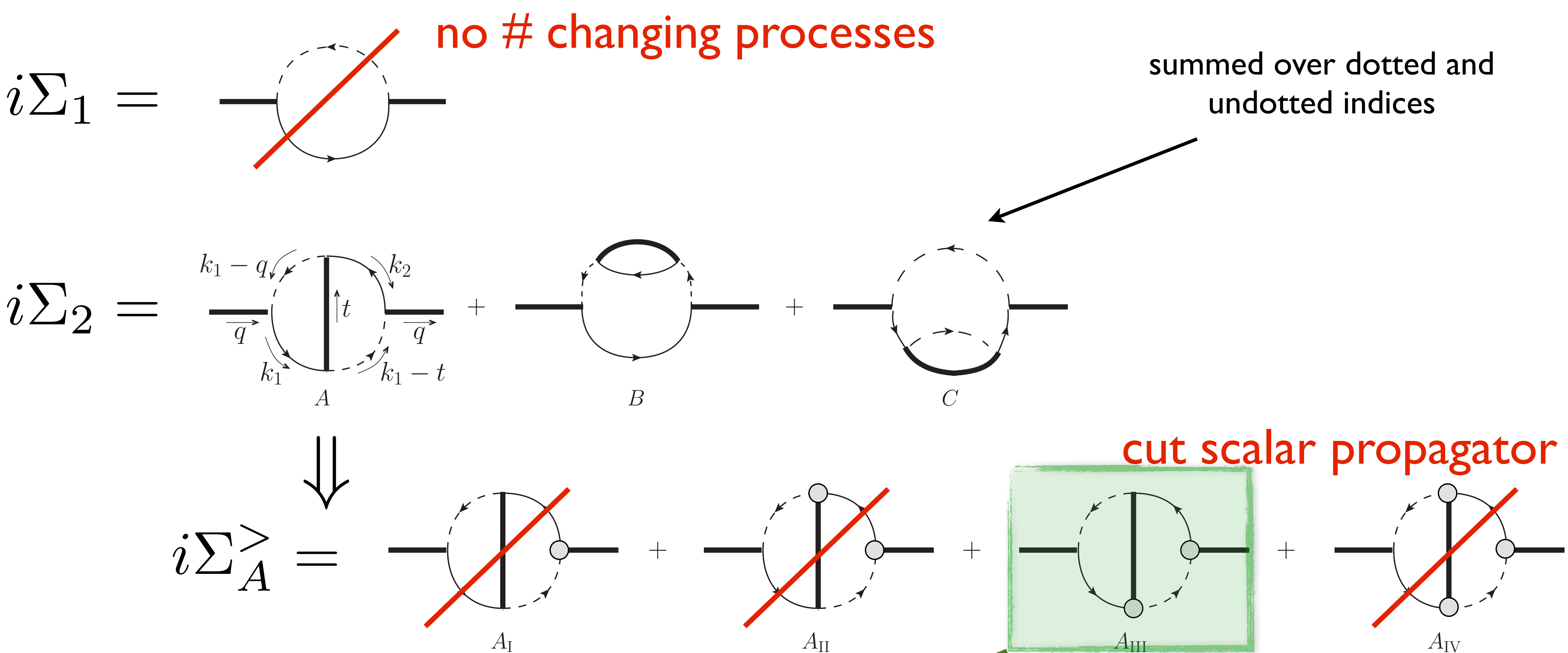
scale hierarchy: $m_\phi \gtrsim m_\chi \gg T \gg m_f$

no thermal contributions

effectively massless

rescaled variables: $\tau = \dfrac{T}{m_\chi} \ll 1$ $\qquad \epsilon = \dfrac{m_f}{2m_\chi} \ll \tau$ $\qquad \xi = \dfrac{m_\phi}{m_\chi} \gtrsim 1$

Collision Term
computation



$$\Sigma^{>}_{A_{\rm III}}(q)\, S^{<}(q) = -\lambda^4 \int \frac{d^4t}{(2\pi)^4}\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}(2\pi)^4\,\delta(q+t-k_1-k_2)$$
$$\underbrace{i\Delta^{11}(k_1-q)\, i\Delta^{22}(k_1-t)}_{\equiv\mathcal{S}}\;\underbrace{P_{\rm R} iS^{21}(k_2)\, P_{\rm L} iS^{12}(t)\, P_{\rm L} iS^{21}(k_1)\, P_{\rm R} iS^{12}(q)}_{\equiv\mathcal{F}}$$

COLLISION TERM

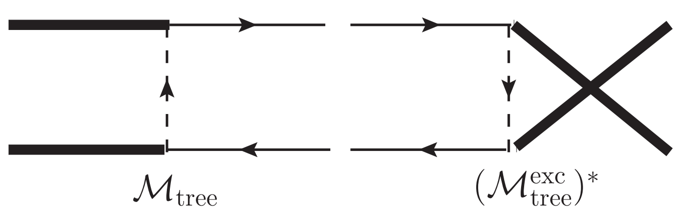
MATCHING

after inserting propagators:

$$\Sigma^{>}_{A_{\rm III}}(q)\, S^{<}(q) = \frac{1}{2E_{\chi_1}}(2\pi)\,\delta\left(q^0 - E_{\chi_1}\right)\int \frac{d^4t}{(2\pi)^3\, 2E_{\chi_2}}\delta\left(t^0 - E_{\chi_2}\right)\times$$

$$\int \frac{d^3\vec{k}_1}{(2\pi)^3\, 2E_{f_1}}\frac{d^3\vec{k}_2}{(2\pi)^3\, 2E_{f_2}}(2\pi)^4\,\delta\left(q + t - k_1 - k_2\right)|\mathcal{M}_A|^2\left[f_\chi(q)\, f_\chi(t)\left(1 - f_f^{\rm eq}\left(k_1^0\right)\right)\left(1 - f_f^{\rm eq}\left(k_2^0\right)\right)\right]$$

$\Rightarrow$ one indeed recovers the known collision term and

$|\mathcal{M}_A|^2 =$ 

(part of) tree level $|\mathcal{M}|^2$

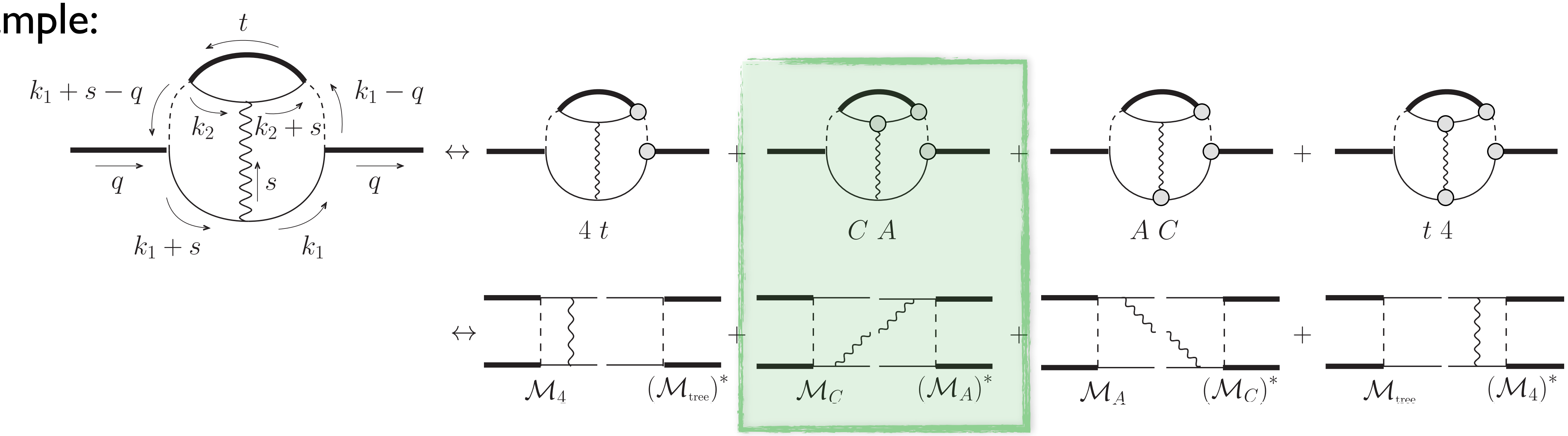
repeating the same for B type diagrams leads to conclusion:

$i\Sigma^{>} \leftrightarrow$ tree level annihilation contribution to the collision term

Collision Term
matching at NLO

$i\Sigma_3 =$ 20 self-energy diagrams

example:



$$\Sigma^{>}_{\text{CA}}(q)\,S^{<}(q) = \frac{1}{2E_{\chi_1}}(2\pi)\,\delta\left(q^0-E_{\chi_1}\right)\int\frac{d^4t}{(2\pi)^3\,2E_{\chi_2}}\delta\left(t^0-E_{\chi_2}\right)$$
$$\int\frac{d^3\vec{k}_1}{(2\pi)^3\,2E_{f_1}}\frac{d^3\vec{k}_2}{(2\pi)^3\,2E_{f_2}}\frac{d^3\vec{s}}{(2\pi)^3\,2E_{\gamma}}(2\pi)^4\,\delta\left(q+t-k_1-k_2-s\right)$$
$$\mathcal{M}_C\left(\mathcal{M}_A\right)^*\left[f_\chi(q)\,f_\chi(t)\left(1-f_f^{\text{eq}}\left(k_1^0\right)\right)\left(1-f_f^{\text{eq}}\left(k_2^0\right)\right)\left(1+f_\gamma^{\text{eq}}\left(s^0\right)\right)\right]$$

$\Rightarrow$ at NLO thermal effects do **not** change the collision therm structure

RESULTS

coming back to our example...

every contribution can be written in a form:

$$\int_0^\infty d\omega\, f_\gamma(\omega)\, S(\omega, e_\chi, \epsilon, \xi) \qquad f_\gamma(\omega) = \frac{1}{1 - e^{\omega/T}}$$

photon energy

$$S = \sum_{n=-1}^{\infty} s_n \omega^n$$

note:

$$J_n \equiv \int_0^\infty f_B(\omega)\omega^n d\omega = \begin{cases} \text{div} & n \leq 0 \\ \sim \tau^{n+1} & n > 0 \end{cases}$$

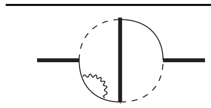
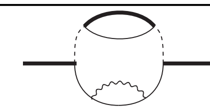
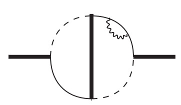
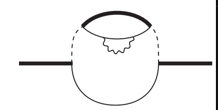
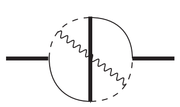
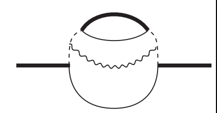
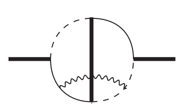
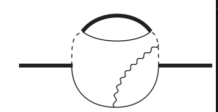
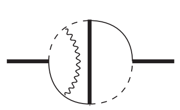
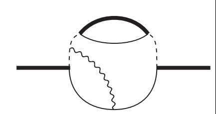
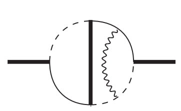
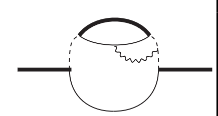
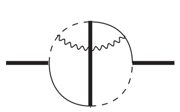
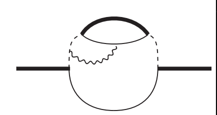
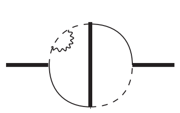
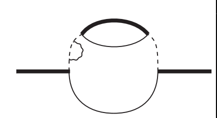
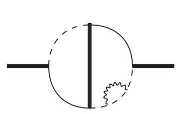
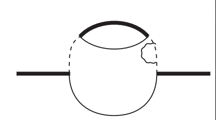
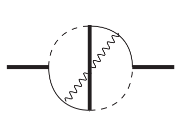
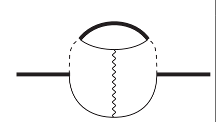
IR divergence in separate terms: $J_{-1} \leftrightarrow T = 0$ soft div
$J_0 \leftrightarrow T = 0$ soft eikonal

finite T correction: $J_1 \leftrightarrow \mathcal{O}(\tau^2)$

Results

IR divergence cancellation: S-wave

The divergent part J_{-1}

Type A	Real	Virtual	External	Type B	Real	Virtual	External
	$\frac{\alpha(1-2\epsilon^2)}{\pi\epsilon^2}$		$-\frac{\alpha(1-2\epsilon^2)}{\pi\epsilon^2}$		$-\frac{\alpha}{\pi\epsilon^2}$		$\frac{\alpha}{\pi\epsilon^2}$
	$\frac{\alpha(1-2\epsilon^2)}{\pi\epsilon^2}$		$-\frac{\alpha(1-2\epsilon^2)}{\pi\epsilon^2}$		$-\frac{\alpha}{\pi\epsilon^2}$		$\frac{\alpha}{\pi\epsilon^2}$
	0				0		
	0	0			0	0	
	0	0			0	0	
	0	0			0	0	
	0	0			0	0	
		0				0	
		0				0	
	$\frac{2\alpha(1-2\epsilon^2)^2}{\pi\epsilon^2\sqrt{1-4\epsilon^2}}L$	$-\frac{2\alpha(1-2\epsilon^2)^2}{\pi\epsilon^2\sqrt{1-4\epsilon^2}}L$			$\frac{2\alpha(1-2\epsilon^2)}{\pi\epsilon^2\sqrt{1-4\epsilon^2}}L$	$-\frac{2\alpha(1-2\epsilon^2)}{\pi\epsilon^2\sqrt{1-4\epsilon^2}}L$	

→ cancels in **every** row separately

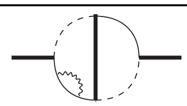
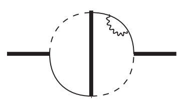
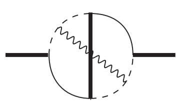
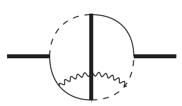
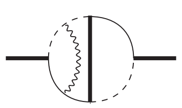
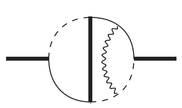
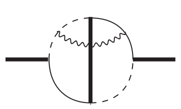
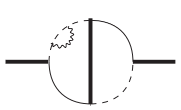
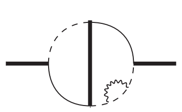
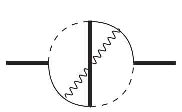
$\Longrightarrow$ every CTP self-energy is IR finite

Results

finite T correction: s-wave

factorized $\frac{\pi}{6}\alpha\tau^2\frac{a_{\text{tree}}}{\epsilon^2}$

The finite part J_1

Type A	Real	Virtual	External
	$\frac{2(1-\xi^2)}{D^2D_\xi^2} + \frac{(1-2\epsilon^2)p_1(\epsilon,\xi)}{2D^2D_\xi^2} + \frac{1}{2\sqrt{D}}L$		$\frac{(1-2\epsilon^2)(\xi^2-3D)}{2DD_\xi} - \frac{1}{2\sqrt{D}}L$
	— " —		— " —
	$-\frac{4(1-2\epsilon^2)D}{D_\xi^2}$		
	$-\frac{2(1-2\epsilon^2)\xi^2}{D_\xi^2} - \frac{f_1(\epsilon,\xi)}{\sqrt{D}D_\xi^2}L$	$\frac{2(1-2\epsilon^2)(D-\xi^2)}{D_\xi^2} + \frac{f_1(\epsilon,\xi)}{\sqrt{D}D_\xi^2}L$	
	— " —	— " —	
	— " —	— " —	
	— " —	— " —	
		$-\frac{4(1-2\epsilon^2)D}{D_\xi^2}$	
		— " —	
	$\frac{2(1-2\epsilon^2)p_2(\epsilon,\xi)+(1-\xi^2)^2}{D^2D_\xi^2} + \frac{4f_2(\epsilon,\xi)}{\sqrt{D}D_\xi^2}L$	$\frac{16\epsilon^2(2-3\epsilon^2)-(3-\xi^2)^2}{D_\xi^2} - \frac{4f_2(\epsilon,\xi)}{\sqrt{D}D_\xi^2}L$	

→ **Log** terms cancels in **every** row separately

⇓

no collinear divergence!

separate contributions complicated... but when summed up:

type A = $-\frac{8(1-2\epsilon^2)}{1+\xi^2-4\epsilon^2}$

type B = $\frac{8}{1+\xi^2-4\epsilon^2}$

Results

finite T correction

Total result for the s-wave:

$$a = a_{\text{tree}}\,(1+\Delta_a) + \mathcal{O}(\tau^4) \qquad \Delta_a = \frac{8\pi}{3}\alpha\tau^2 \frac{1}{1-4\epsilon^2+\xi^2}.$$

$$\xi = \frac{m_\phi}{m_\chi} \gtrsim 1$$

$$\tau = \frac{T}{m_\chi} \ll 1$$

$$\epsilon = \frac{m_f}{2m_\chi} \ll \tau$$

In general corrections from thermal photons:

$$\sigma v = \sigma v_{\text{tree}} - \frac{4}{3}\pi\alpha\tau^2 \frac{\partial}{\partial \xi^2}\sigma v_{\text{tree}} + \mathcal{O}(\tau^4),$$

The helicity suppression is lifted at 4th order in temperature:

$$\Delta a_{\tau^4}^{\epsilon=0} = \frac{8\pi^2\lambda^4\alpha\tau^4}{45}\frac{1}{(1+\xi^2)^4} = \frac{4\pi}{45}\alpha\tau^4 \frac{1}{(1+\xi^2)^2}\frac{a_{\text{tree}}}{\epsilon^2}\bigg|_{\epsilon=0}$$

strongly suppressed as at kinetic equilibrium $\tau \sim v^2$

Conclusions

1. how the (soft and collinear) IR divergence cancellation happen?
   automatic in thermal QFT formalism, cancellation at the level of every CTP self-energy

2. does Boltzmann equation itself receive quantum corrections?
   no, not at NLO

3. how large are the remaining finite T corrections?
   strongly suppressed, of order $\mathcal{O}(\alpha\tau^2)$

Takeaway:

the Boltzmann eq. is safe at NLO...

...but interesting physics awaits along the path to find out why

Backup slides

RELIC DENSITY:
THE LO COLLISION TERM

for $2 \leftrightarrow 2$ CP invariant process:

note: added "by hand"

$$C_{\rm LO} = -h_\chi^2 \int \frac{d^3\vec{p}_\chi}{(2\pi)^3} \frac{d^3\vec{p}_{\bar\chi}}{(2\pi)^3}\, \sigma_{\chi\bar\chi\to ij} v_{\rm rel}\, \left[f_\chi f_{\bar\chi} (1\pm f_i)(1\pm f_j) - f_i f_j (1\pm f_\chi)(1\pm f_{\bar\chi}) \right]$$

assuming kinetic equilibrium at chemical decoupling: $f_\chi \sim a(\mu) f_\chi^{\rm eq}$

$$C_{\rm LO} = -\langle \sigma_{\chi\bar\chi\to ij} v_{\rm rel}\rangle^{\rm eq} \left(n_\chi n_{\bar\chi} - n_\chi^{\rm eq} n_{\bar\chi}^{\rm eq} \right)$$

where the thermally averaged cross section:

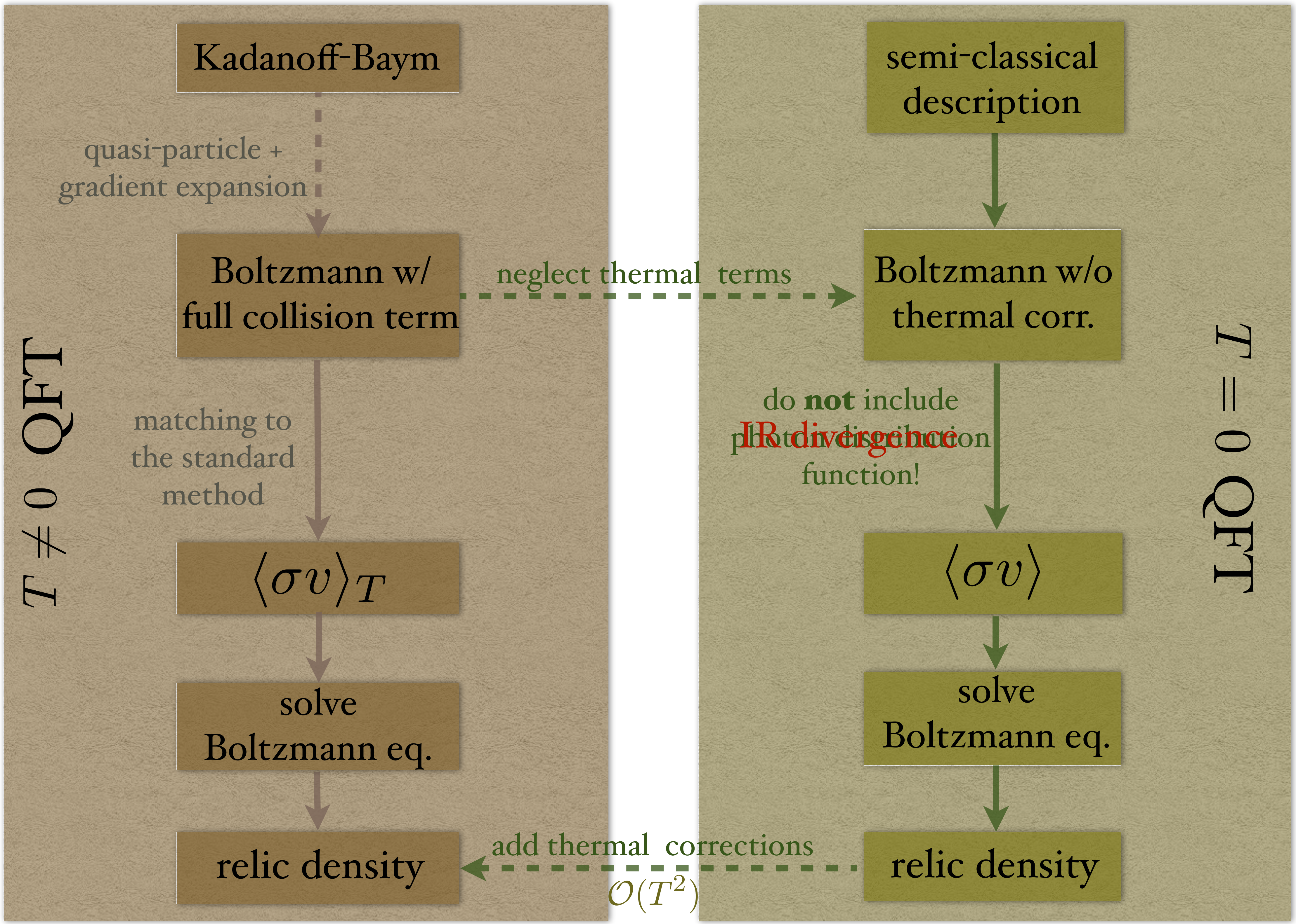
$$\langle \sigma_{\chi\bar\chi\to ij} v_{\rm rel}\rangle^{\rm eq} = -\frac{h_\chi^2}{n_\chi^{\rm eq} n_{\bar\chi}^{\rm eq}} \int \frac{d^3\vec{p}_\chi}{(2\pi)^3} \frac{d^3\vec{p}_{\bar\chi}}{(2\pi)^3}\, \sigma_{\chi\bar\chi\to ij} v_{\rm rel}\, f_\chi^{\rm eq} f_{\bar\chi}^{\rm eq}$$

crucial point: $p_\chi + p_{\bar\chi} = p_i + p_j \Rightarrow f_\chi^{\rm eq} f_{\bar\chi}^{\rm eq} \approx f_i^{\rm eq} f_j^{\rm eq}$

in Maxwell approx.

COLLISION TERM

METHOD SUMMARY



Relic Density:

what happens at NLO?

only this used in NLO literature so far

$$C_{\rm NLO} \sim \int d\Pi_{\chi\bar\chi ij}\; f_\chi f_{\bar\chi} \left\{ |\mathcal{M}^{\rm LO}_{\chi\bar\chi\to ij}|^2 + |\mathcal{M}^{{\rm NLO}\; T=0}_{\chi\bar\chi\to ij}|^2 + \int d\Pi_\gamma |\mathcal{M}_{\chi\bar\chi\to ij\gamma}|^2 + \right.$$

$$|\mathcal{M}^{{\rm NLO}\; T\neq 0}_{\chi\bar\chi\to ij}|^2 + \int d\Pi_\gamma \left[f_\gamma \left(|\mathcal{M}_{\chi\bar\chi\to ij\gamma}|^2 + |\mathcal{M}_{\chi\bar\chi\gamma\to ij}|^2 \right) \right.$$

$$\left.\left. - f_i \left(|\mathcal{M}_{\chi\bar\chi\to ij\gamma}|^2 + |\mathcal{M}_{\chi\bar\chi i\to j\gamma}|^2 \right) - f_j \left(|\mathcal{M}_{\chi\bar\chi\to ij\gamma}|^2 + |\mathcal{M}_{\chi\bar\chi j\to i\gamma}|^2 \right) \right] \right\}$$

thermal 1-loop

photon absorption

photon emission

SM fermions emission

SM fermions absorption

$$-f_i f_j \left\{ |\mathcal{M}^{\rm LO}_{ij\to\chi\bar\chi}|^2 + |\mathcal{M}^{{\rm NLO}\; T=0}_{ij\to\chi\bar\chi}|^2 + \int d\Pi_\gamma |\mathcal{M}_{ij\to\chi\bar\chi\gamma}|^2 + \right.$$

$$|\mathcal{M}^{{\rm NLO}\; T\neq 0}_{ij\to\chi\bar\chi}|^2 + \int d\Pi_\gamma \left[f_\gamma \left(|\mathcal{M}_{ij\to\chi\bar\chi\gamma}|^2 + |\mathcal{M}_{ij\gamma\to\chi\bar\chi}|^2 \right) \right.$$

$$\left.\left. - f_\chi \left(|\mathcal{M}_{ij\to\chi\bar\chi\gamma}|^2 + |\mathcal{M}_{ij\chi\to\chi\gamma}|^2 \right) - f_{\bar\chi} \left(|\mathcal{M}_{ij\to\chi\bar\chi\gamma}|^2 + |\mathcal{M}_{ij\bar\chi\to\bar\chi\gamma}|^2 \right) \right] \right\}$$

the production and annihilation in general differ:

$$C_{\rm NLO} = -\left[\langle \sigma^{\rm NLO}_{\rm ann}\, v_{\rm rel} \rangle^{\rm eq}\, n_\chi n_{\bar\chi} - \langle \sigma^{\rm NLO}_{\rm prod}\, v_{\rm rel} \rangle^{\rm eq}\, n^{\rm eq}_\chi n^{\rm eq}_{\bar\chi} \right]$$