(More) exceptions
in the calculations of Relic Abundances

Andrzej Hryczuk

University of Oslo

based on: **T. Binder, T. Bringmann, M. Gustafson and AH** (to appear soon)

(if enough time also **M. Beneke, F. Dighera, AH**, 1409.3049, 1607.03910)

Stavanger, 20th June 2017

Thermal Relic Density

"Exceptions"

1. Co-annihilations, thresholds and poles

   Griest, Seckel '91

2. Bound State Formation

   recent e.g., Petraki at al. '15, '16; An et al. '15, '16; Cirelli et al. '16; …

3. $3 \rightarrow 2$ and $4 \rightarrow 2$ annihilation

   e.g., D'Agnolo, Ruderman '15; Cline at al. '17; Choi at al. '17; …

4. Semi-annihilation

   D'Eramo, Thaler '10

5. Finite temperature effects

   Wizansky '06; Beneke, Dighera, AH '14, '16

6. Non-standard cosmology

   many works… very recent e.g., D'Eramo, Fernandez, Profumo '17

7. …

Many of these "exceptions" appear for non-minimal scenarios and do have significant impact — but do not affect the foundations of modern calculations

Outline

1. Introduction
    - standard approach to thermal relic density

2. Exception ***n***
    - early kinetic decoupling with
    - velocity dependent annihilation

3. Exception ***n+1***
    - NLO effects at
    - finite temperature

4. Summary

THERMAL RELIC DENSITY

STANDARD APPROACH

T

$\Gamma_{\rm ann} > H$ DM in equilibrium

$\Gamma_{\rm ann} \sim H$ chemical decoupling time

$\Gamma_{\rm ann} < H$ freeze-out

non-equilibrium

time evolution of $f_\chi(p)$ in kinetic theory:

$$\boxed{E\left(\partial_t - H\vec{p}\cdot\nabla_{\vec{p}}\right)} f_\chi = \mathcal{C}[f_\chi] \quad \Longrightarrow \quad \frac{dn_\chi}{dt} + 3Hn_\chi = C$$

Liouville operator in FRW background

the collision term

integrated

assumptions for using Boltzmann eq: classical limit, molecular chaos,...

(for derivation from thermal QFT… we will come back to it later)

Thermal Relic Density

THE COLLISION TERM

for $2 \leftrightarrow 2$ CP invariant process:

$$C_{\mathrm{LO}} = -h_\chi^2 \int \frac{d^3\vec{p}_\chi}{(2\pi)^3} \frac{d^3\vec{p}_{\bar{\chi}}}{(2\pi)^3}\, \sigma_{\chi\bar{\chi}\to ij} v_{\mathrm{rel}}\, \left[f_\chi f_{\bar{\chi}}(1 \pm f_i)(1 \pm f_j) - f_i f_j (1 \pm f_\chi)(1 \pm f_{\bar{\chi}})\right]$$

assuming kinetic equilibrium at chemical decoupling: $f_\chi \sim a(\mu) f_\chi^{\mathrm{eq}}$

$$\boxed{C_{\mathrm{LO}} = -\langle \sigma_{\chi\bar{\chi}\to ij} v_{\mathrm{rel}} \rangle^{\mathrm{eq}} \left(n_\chi n_{\bar{\chi}} - n_\chi^{\mathrm{eq}} n_{\bar{\chi}}^{\mathrm{eq}}\right)}$$

where the thermally averaged cross section:

$$\langle \sigma_{\chi\bar{\chi}\to ij} v_{\mathrm{rel}} \rangle^{\mathrm{eq}} = -\frac{h_\chi^2}{n_\chi^{\mathrm{eq}} n_{\bar{\chi}}^{\mathrm{eq}}} \int \frac{d^3\vec{p}_\chi}{(2\pi)^3} \frac{d^3\vec{p}_{\bar{\chi}}}{(2\pi)^3}\, \sigma_{\chi\bar{\chi}\to ij} v_{\mathrm{rel}}\, f_\chi^{\mathrm{eq}} f_{\bar{\chi}}^{\mathrm{eq}}$$

Thermal Relic Density

Boltzmann eq.

$$\frac{dn_\chi}{dt} + 3Hn_\chi = -\langle \sigma_{\chi\bar{\chi}\to ij}\sigma_{\rm rel}\rangle^{\rm eq}\left(n_\chi n_{\bar\chi} - n_\chi^{\rm eq} n_{\bar\chi}^{\rm eq}\right)$$

Re-written for the comoving number density:

$$\frac{dY}{dx} = \sqrt{\frac{g_* \pi m_\chi^2}{45G}}\,\frac{\langle \sigma_{\chi\bar{\chi}\to ij}\sigma_{\rm rel}\rangle^{\rm eq}}{x^2}\left(Y^2 - Y_{\rm eq}^2\right)$$

$$\lim_{x\to 0} Y = Y_{\rm eq} \qquad \lim_{x\to\infty} Y = \text{const}$$

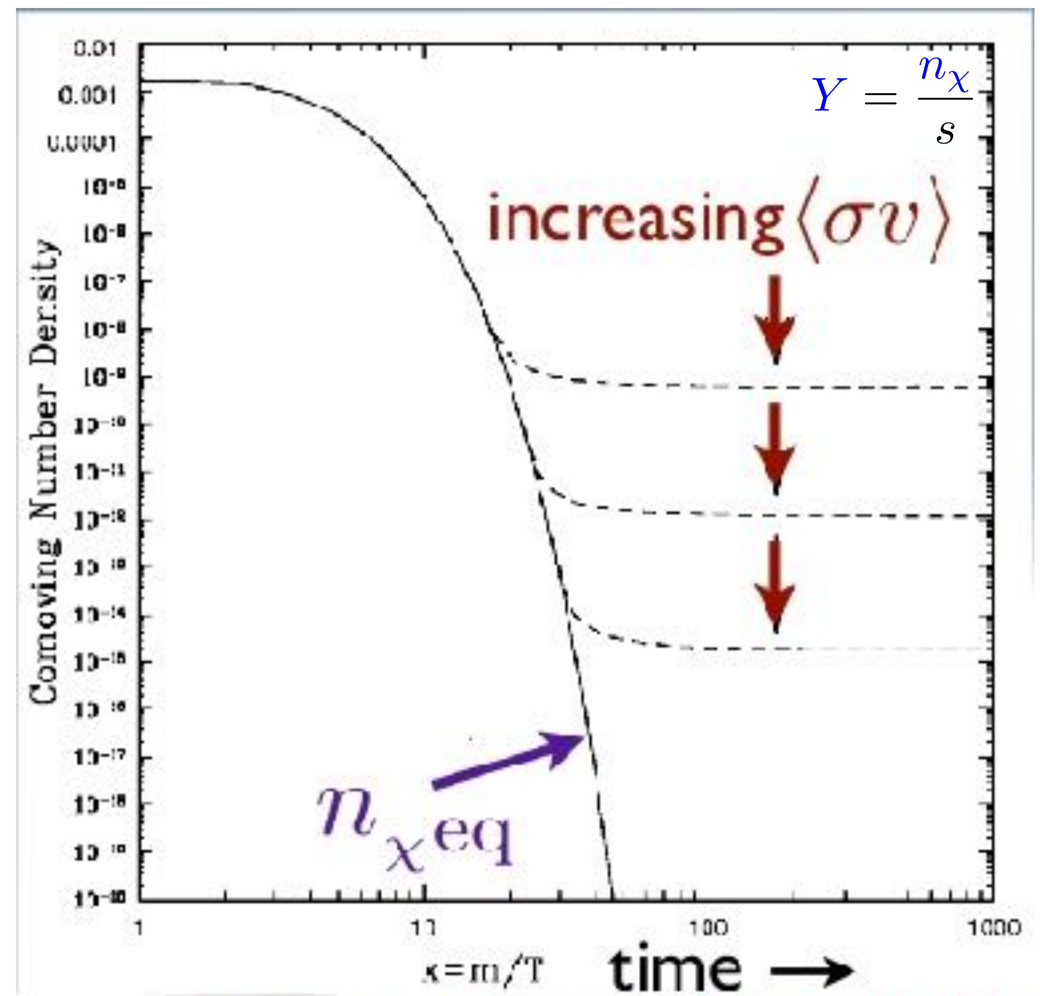


Fig.: Jungman, Kamionkowski & Griest, PR'96

Recipe:
compute annihilation cross-section,
take a thermal bath average,
throw it into BE… and voilà

Exception *n*:
Early kinetic decoupling

A Pitfall in a Nutshell

T

$\Gamma_{\rm ann} > H$ DM in equilibrium

$\Gamma_{\rm ann} \sim H$ chemical decoupling

$\Gamma_{\rm ann} < H$ freeze-out

time

can it happen?

typically

kinetic decoupling

If KD happens before CD $\longrightarrow$ what would be the relic density?

how to even compute that? $\Longrightarrow$ need for refined treatment of solving the Boltzmann eq.

assuming kinetic equilibrium at chemical decoupling: $f_\chi \sim a(\mu) f_\chi^{\rm eq}$

$$C_{\rm LO} = -\langle \sigma_{\chi\bar\chi\to ij} v_{\rm rel}\rangle^{\rm eq} \left(n_\chi n_{\bar\chi} - n_\chi^{\rm eq} n_{\bar\chi}^{\rm eq}\right)$$

Kinetic Decoupling 101

We start from full BE: $$E\left(\partial_t - H\vec{p}\cdot\nabla_{\vec{p}}\right) f_\chi = \mathcal{C}[f_\chi]$$

contains both scatterings and annihilation

First consider only temperature evolution - i.e. leave out feedback **on** number density, and *define*:

$$T_\chi \equiv \frac{g_\chi}{3m_\chi n_\chi}\int\frac{\mathrm{d}^3p}{(2\pi)^3}p^2 f_\chi(p) \qquad\qquad y \equiv \frac{m_\chi T_\chi}{s^{2/3}}$$

then 2nd moment of full BE (up to terms p^2/m_χ^2) gives:

$$\boxed{\frac{y'}{y} = \qquad - \left(1-\frac{x}{3}\frac{g_{*S}'}{g_{*S}}\right)\frac{2m_\chi c(T)}{Hx}\left(1-\frac{y_{\rm eq}}{y}\right)}$$

Bringmann, Hofmann JCAP 0704 (2007) 016

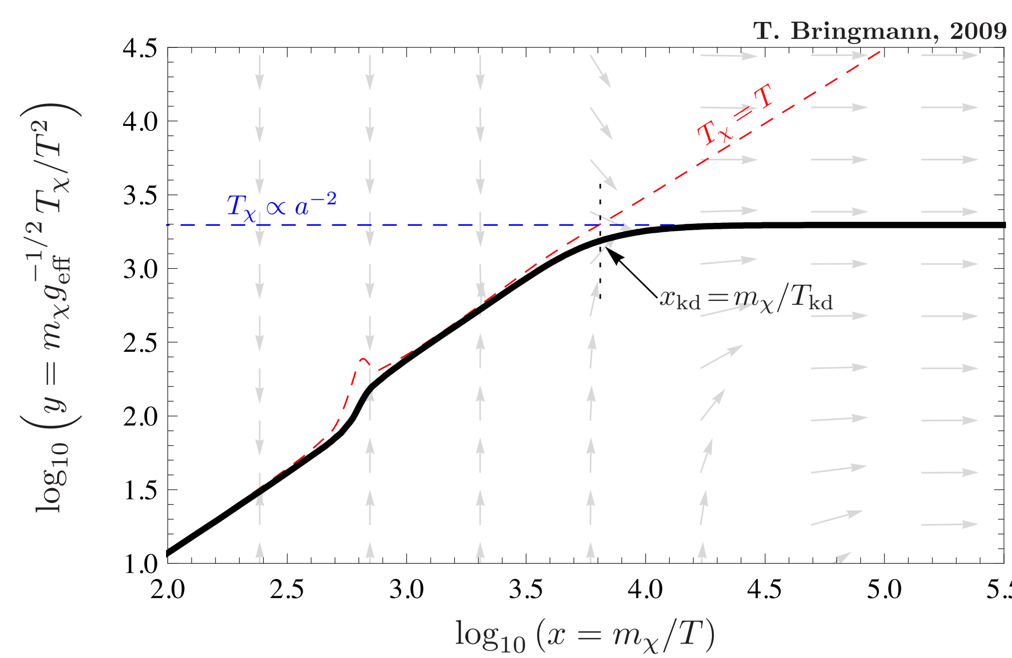
where:

$$\langle\sigma v_{\rm rel}\rangle_2 \quad\equiv\quad \frac{g_\chi^2}{3Tm_\chi n_\chi^2}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3\tilde{p}}{(2\pi)^3}\,p^2 v_{\rm rel}\sigma_{\bar{\chi}\chi\to\bar{X}X} f(E)f(\tilde{E})$$

impact of annihilation

$$c(T) = \frac{1}{12(2\pi)^3 m_\chi^4 T}\sum_X\int dk\,k^5\omega^{-1}\,g^\pm\left(1\mp g^\pm\right)\int_{-4k^2}^{0}(-t)\frac{1}{8k^4}|\mathcal{M}_{\rm el}|^2$$

impact of elastic scatterings



ONE STEP FURTHER…

The full evolution of DM temperature and number density is governed by a coupled system of BEs for 0th and 2nd moments:

annihilation and production thermal averages done at different T — feedback of modified y evolution

$$\frac{Y'}{Y} = -\frac{1-\frac{x}{3}\frac{g'_{*S}}{g_{*S}}}{Hx} sY \left(\langle\sigma v_{\rm rel}\rangle|_{x=m_\chi^2/(s^{2/3}y)} - \frac{Y_{\rm eq}^2}{Y^2}\langle\sigma v_{\rm rel}\rangle|_x \right)$$

$$\frac{y'}{y} = -\frac{1-\frac{x}{3}\frac{g'_{*S}}{g_{*S}}}{Hx}\left[2m_\chi c(T)\left(1-\frac{y_{\rm eq}}{y}\right) - sY\left(\left(\langle\sigma v_{\rm rel}\rangle - \langle\sigma v_{\rm rel}\rangle_2\right)_{x=m_\chi^2/(s^{2/3}y)} - \frac{Y_{\rm eq}^2}{Y^2}\left(\langle\sigma v_{\rm rel}\rangle - \frac{y_{\rm eq}}{y}\langle\sigma v_{\rm rel}\rangle_2\right)_x \right)\right]$$

$$-\frac{4H}{3\sqrt{\pi}x\tilde{H}}\sum_{m=1}^{\infty}(-1)^m\Gamma\left(m+\frac{5}{2}\right)\left(\frac{2}{x}\right)^m\frac{K_{m+2}(x)}{K_2(x)}$$

"relativistic" term

$$T_\chi \equiv \frac{g_\chi}{3n_\chi}\int\frac{d^3p}{(2\pi)^3}\frac{p^2}{E}f_\chi(p)$$

elastic scatterings term

impact of annihilation

<u>These equations still assume the equilibrium shape of $f_\chi(p)$ — but with variant temperature</u>

Example:
Scalar Signlet DM

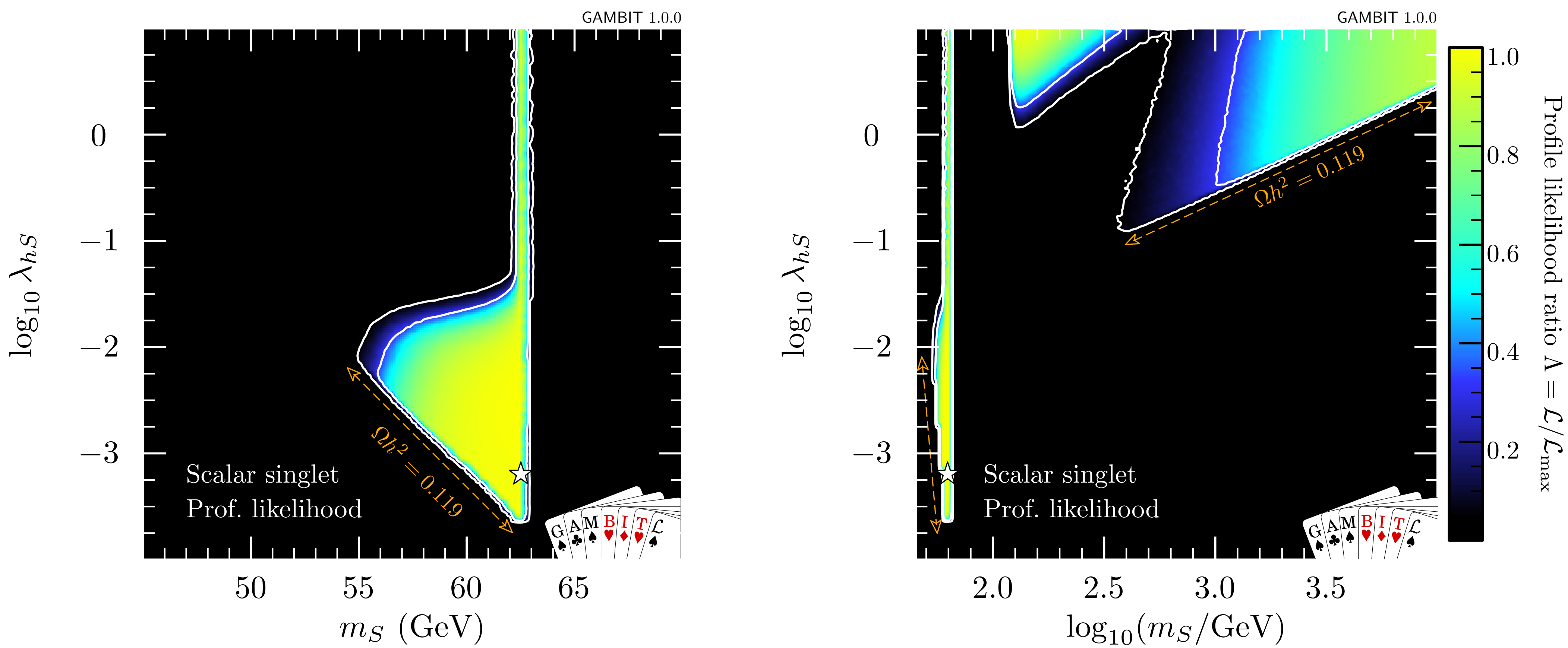
SCALAR SINGLET DM

VERY SHORT INTRODUCTION

To the SM Lagrangian add one singlet scalar field *S* with interactions with the Higgs:

$$\mathcal{L}_S = \frac{1}{2}\partial_\mu S \partial^\mu S - \frac{1}{2}\mu_S^2 S^2 - \frac{1}{2}\lambda_s S^2 |H|^2$$

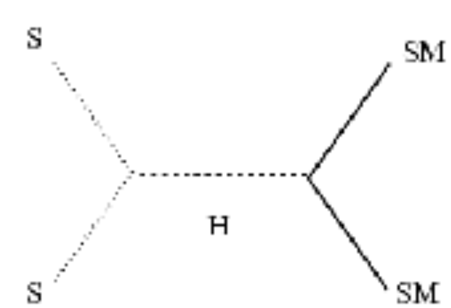
$$m_s = \sqrt{\mu_S^2 + \frac{1}{2}\lambda_s v_0^2}$$



GAMBIT collaboration
1705.07931

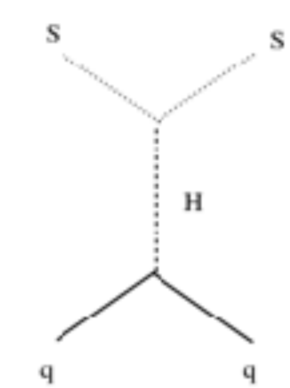
Annihilation processes:

resonant



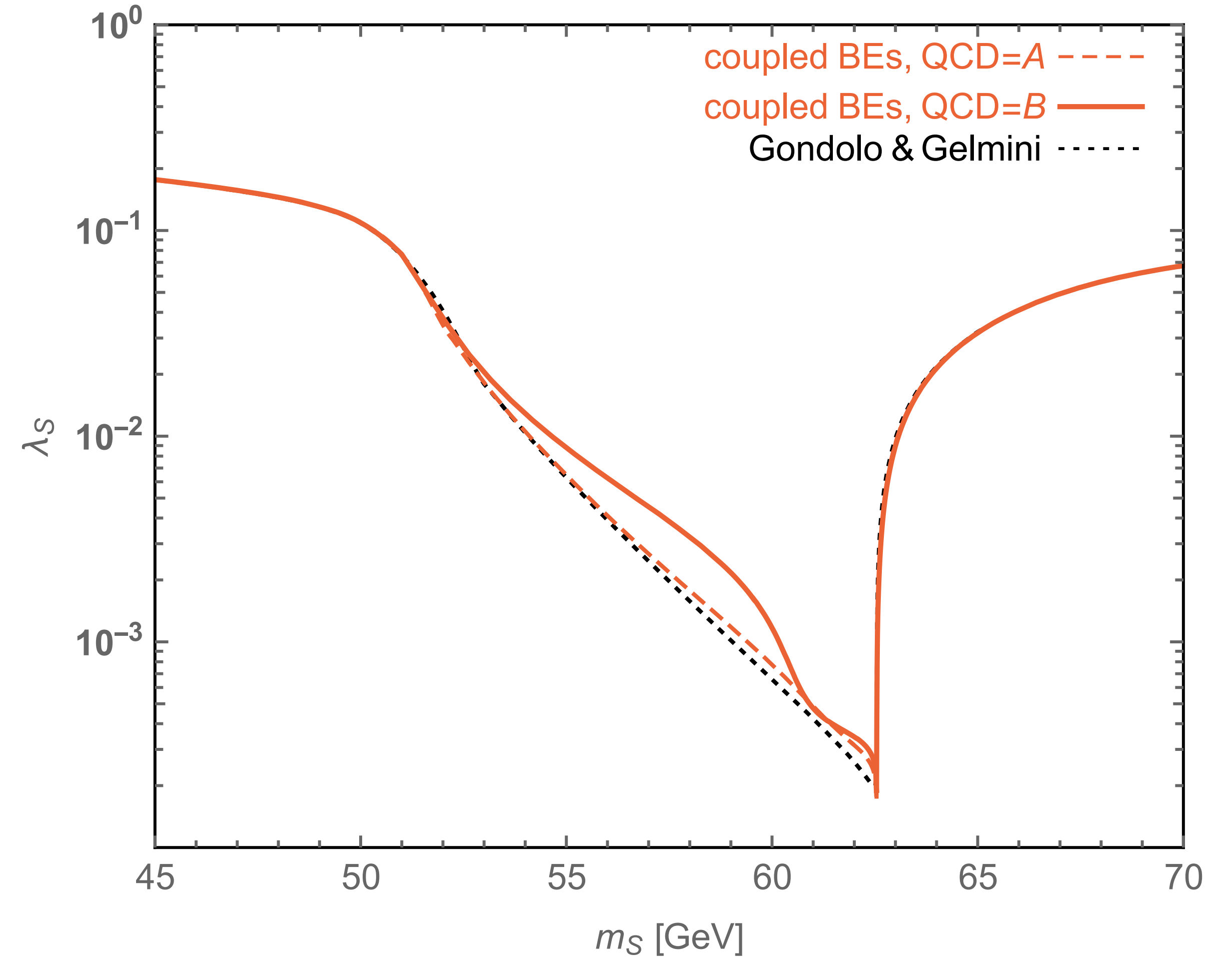
El. scattering processes:

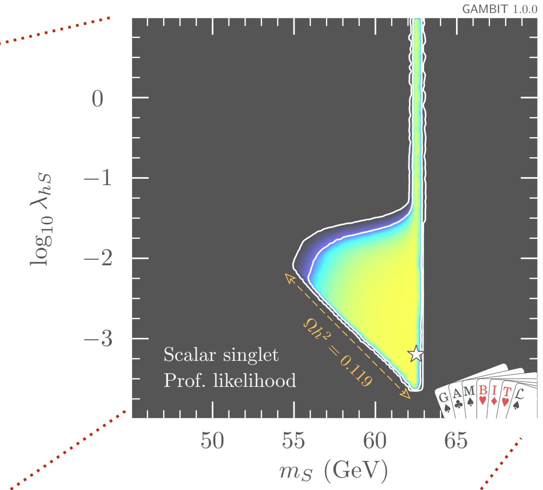
non-resonant



RESULTS

RD CONTOURS





essentially the only region left for this model

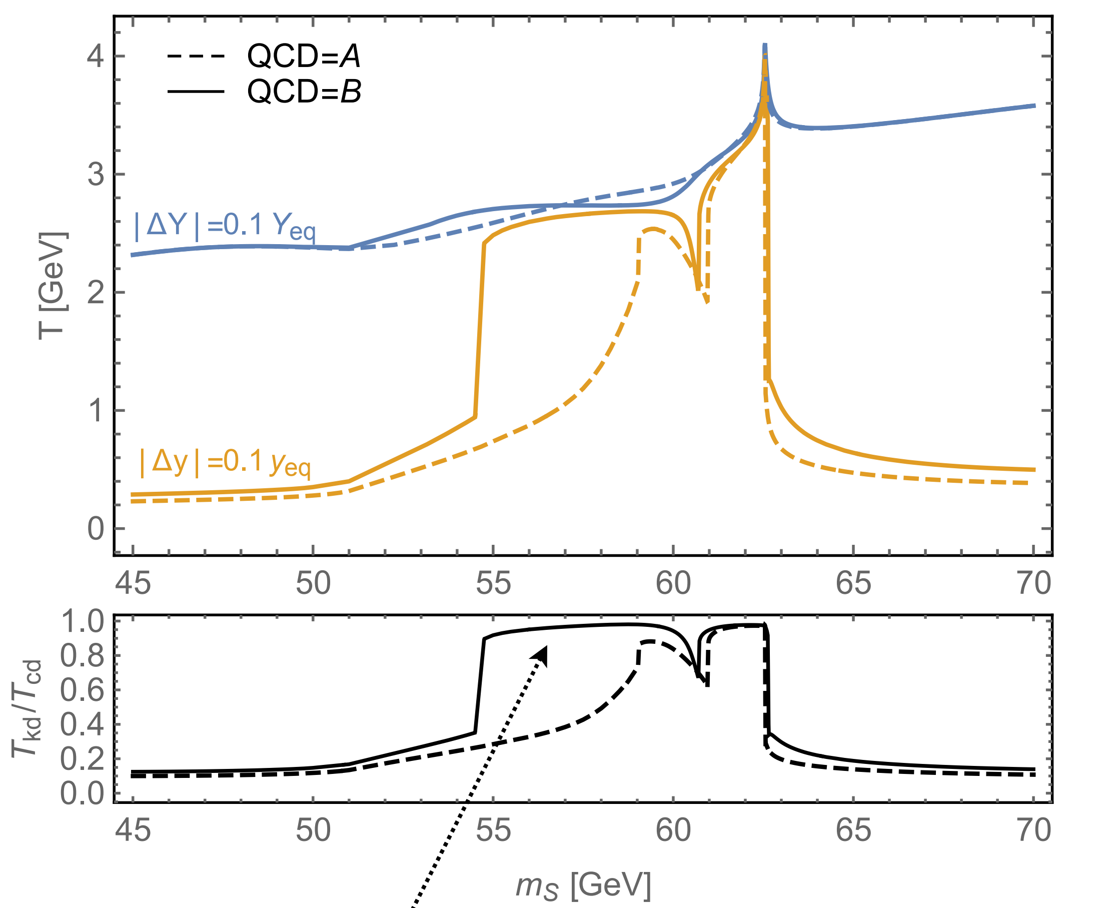
Significant modification of the observed relic density contour in the Scalar Singlet DM model

→ larger coupling needed → better chance for closing the last window

Results

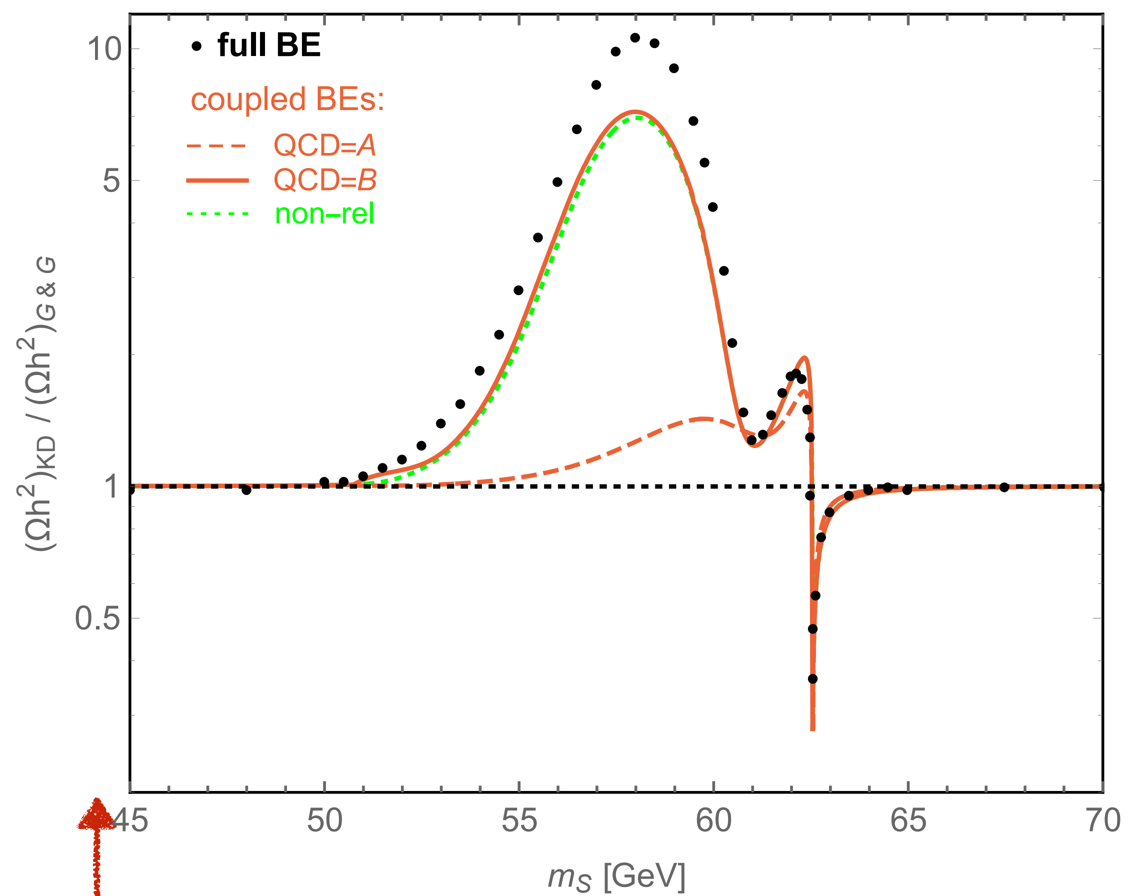
Effect

kinetic and chemical decoupling:



approaches 1, but
does not reach it!

effect on relic density:



effect on relic density:
up to O(~10)

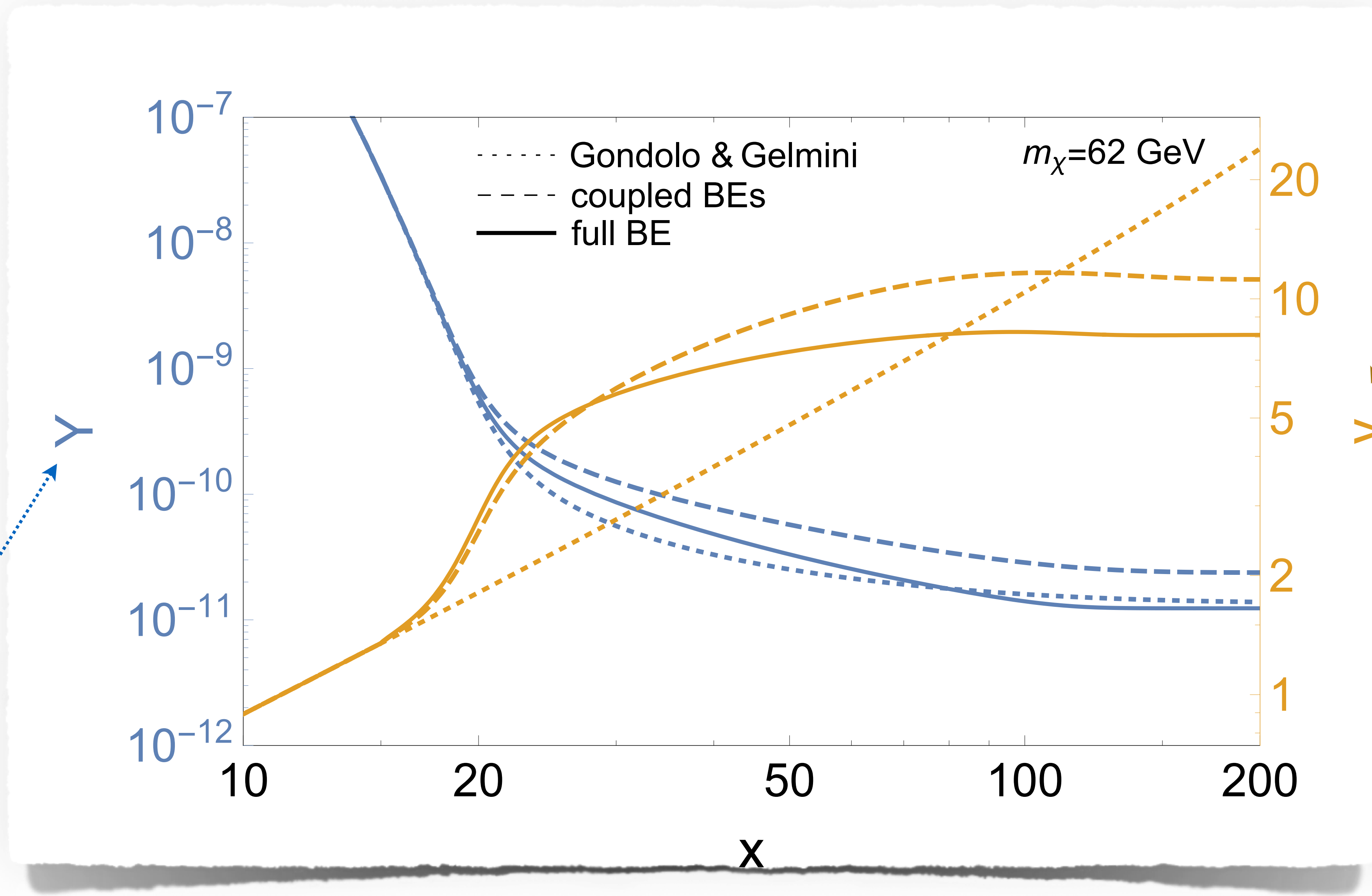
Why such non-trivial shape of the effect of early kinetic decoupling?

Let's inspect the y and Y evolution…

DENSITY AND T_{DM} EVOLUTION

for m_{DM} = 62 GeV,   i.e. just below the resonance:

DM temperature



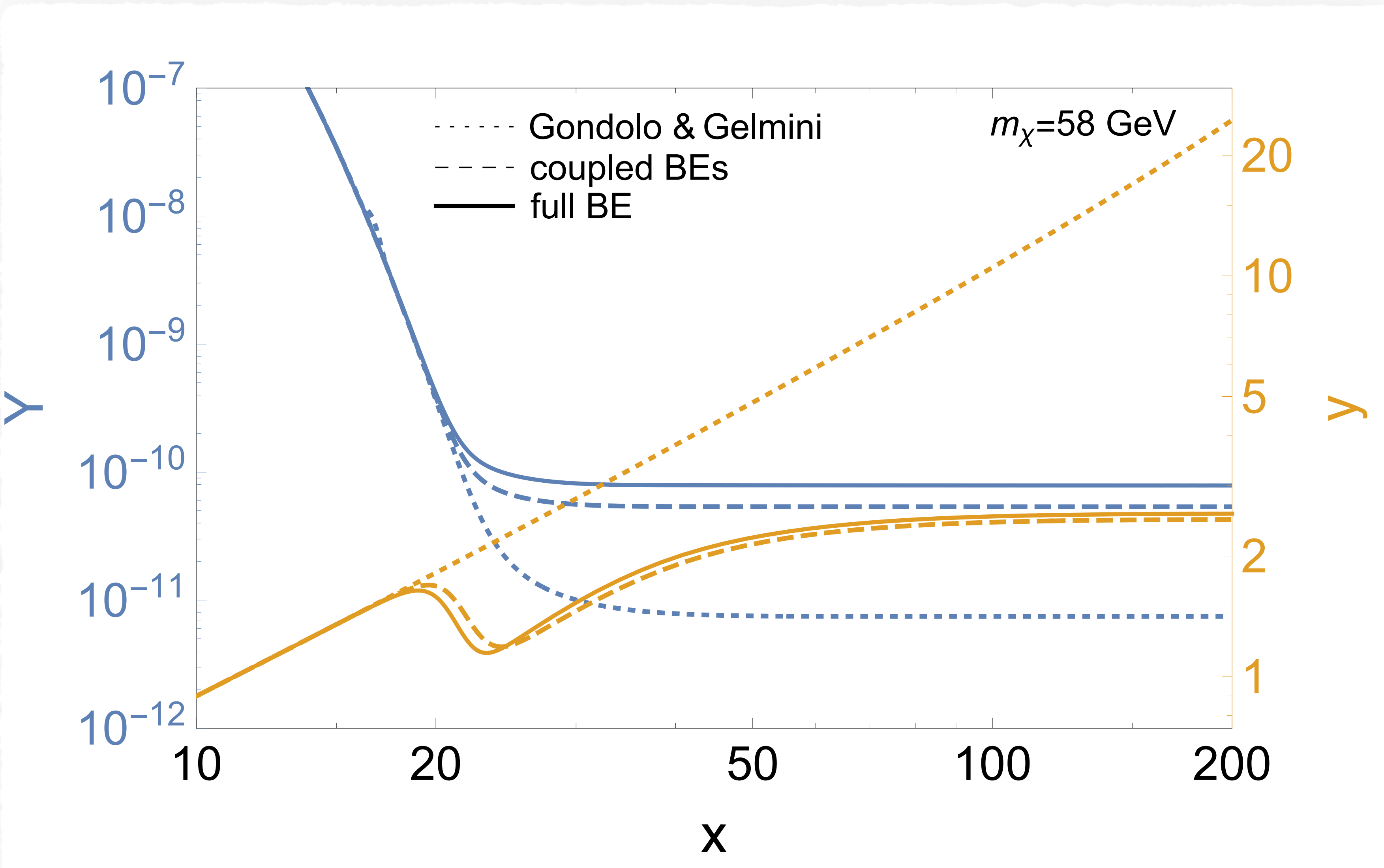
co-moving number density

Resonant annihilation **most effective** for **low momenta**

⟶ DM fluid goes through "heating" phase before leaves kinetic equilibrium

DENSITY AND T_{DM} EVOLUTION

for m_{DM} = 58 GeV, i.e. further away from the resonance:



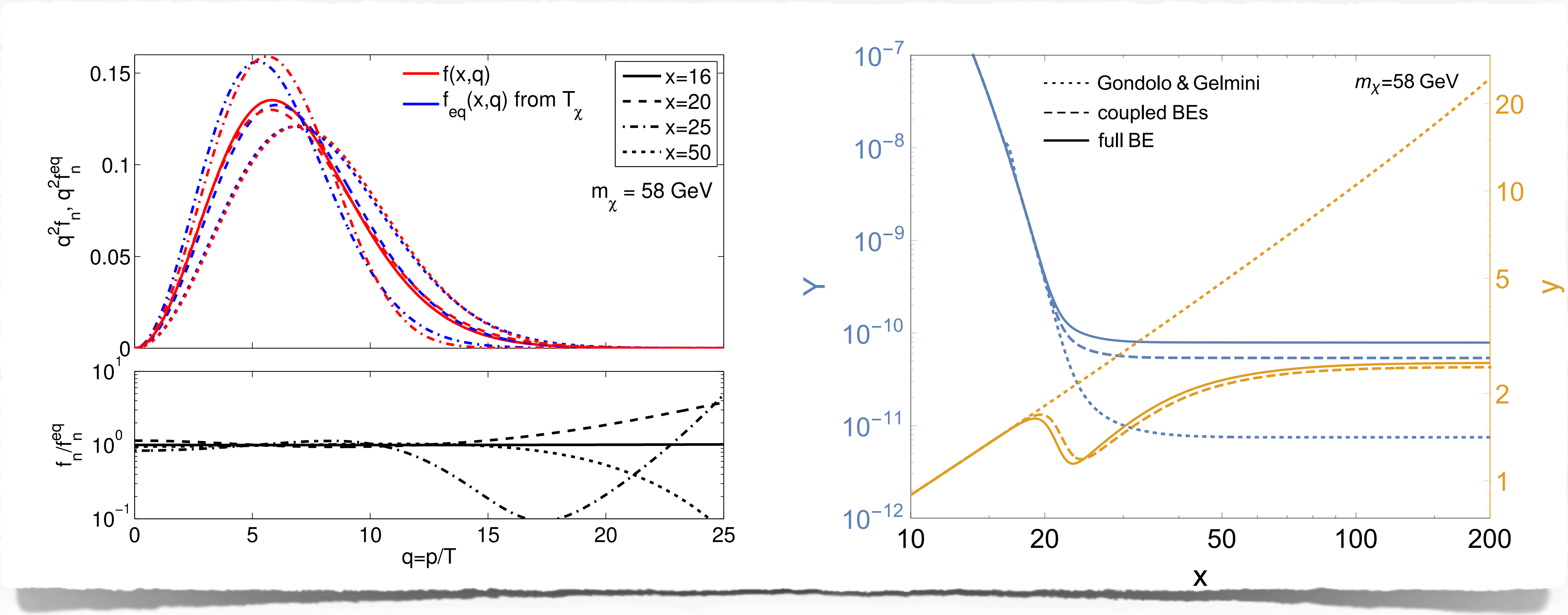
Resonant annihilation **most effective** for **high momenta**

⟶ DM fluid goes through fast "cooling" phase

after that when T_{DM} drops to much annihilation not effective anymore

Full phase-space BE solver

Solutions for full phase-space distribution function:



The numerical approach based on discretization in momentum and solving system of coupled differential equations

Allows to study the evolution of $f_\chi(p)$ and
the interplay between scatterings and annihilation!

SUMMARY: PART I

1. One needs to remember that kinetic equilibrium is a necessary assumption for standard relic density calculations

2. Coupled system of Boltzmann equations for 0th and 2nd moments allow for a very accurate treatment of the kinetic decoupling and its effect on relic density

3. In special cases the full phase space Boltzmann equation can be necessary — especially if one wants to trace DM temperature as well

Exception *N*:
sometimes kinetic decoupling happens very early…

EXCEPTION *n*+1:

NLO EFFECTS

Dark Matter at NLO

Bergstrom '89; Drees et al., 9306325; Ullio & Bergstrom, 9707333 — helicity suppression lifting

⋮

Bergstrom et al., 0507229; Bringmann et al., 0710.3169 — spectral features in indirect searches

⋮

Ciafaloni et al., 1009.0224
Cirelli et al., 1012.4515
Ciafaloni et al., 1202.0692
AH & Iengo, 1111.2916
— large EW corrections

⋮

Chatterjee et al., 1209.2328
Harz et al., 1212.5241
Ciafaloni et al., 1305.6391
Hermann et al., 1404.2931
Boudjema et al., 1403.7459
— ***thermal relic density***

$$\Omega_{DM} h^2 = 0.1187 \pm 0.0017.$$

<1.5% uncertainty!

Planck+WMAP pol.+highL+BAO; 1303.5062

⋮

SloopS, DM@NLO, PPC4DMID — NLO codes

Relic Density at NLO

Recall at LO:

$$C_{\rm LO} = -h_\chi^2 \int \frac{d^3\vec{p}_\chi}{(2\pi)^3}\frac{d^3\vec{p}_{\bar\chi}}{(2\pi)^3}\ \sigma_{\chi\bar\chi\to ij} v_{\rm rel}\ \left[f_\chi f_{\bar\chi}(1\pm f_i)(1\pm f_j) - f_i f_j (1\pm f_\chi)(1\pm f_{\bar\chi})\right]$$

crucial point: $p_\chi + p_{\bar\chi} = p_i + p_j \Rightarrow f_\chi^{\rm eq} f_{\bar\chi}^{\rm eq} \approx f_i^{\rm eq} f_j^{\rm eq}$

in Maxwell approx.

at NLO both virtual one-loop and 3-body processes contribute:

$$C_{1-{\rm loop}} = -h_\chi^2 \int \frac{d^3\vec{p}_\chi}{(2\pi)^3}\frac{d^3\vec{p}_{\bar\chi}}{(2\pi)^3}\ \sigma^{1-{\rm loop}}_{\chi\bar\chi\to ij} v_{\rm rel}\ \left[f_\chi f_{\bar\chi}(1\pm f_i)(1\pm f_j) - f_i f_j (1\pm f_\chi)(1\pm f_{\bar\chi})\right]$$

$$C_{\rm real} = -h_\chi^2 \int \frac{d^3\vec{p}_\chi}{(2\pi)^3}\frac{d^3\vec{p}_{\bar\chi}}{(2\pi)^3}\ \sigma_{\chi\bar\chi\to ij\gamma} v_{\rm rel}\ \left[f_\chi f_{\bar\chi}(1\pm f_i)(1\pm f_j)(1+f_\gamma) - f_i f_j f_\gamma (1\pm f_\chi)(1\pm f_{\bar\chi})\right]$$

$$p_\chi + p_{\bar\chi} = p_i + p_j \pm p_\gamma \Rightarrow$$

photon can be arbitrarily soft

$f_\gamma \sim \omega^{-1}$

Maxwell approx. not valid anymore...

Relic Density at NLO

...problem: *T*-dependend IR divergence!

it sounds scary - but somehow we all know there has to be a happy-end

Relic Density

what Really happens at NLO?

Beneke, Dighera, AH, 1409.3049

only this used in NLO literature so far

$$C_{\rm NLO} \sim \int d\Pi_{\chi\bar\chi ij} \;\; f_\chi f_{\bar\chi} \; \boxed{\Big\{ |\mathcal{M}^{\rm LO}_{\chi\bar\chi\to ij}|^2 + |\mathcal{M}^{{\rm NLO}\;T=0}_{\chi\bar\chi\to ij}|^2 + \int d\Pi_\gamma |\mathcal{M}_{\chi\bar\chi\to ij\gamma}|^2 +}$$

$$|\mathcal{M}^{{\rm NLO}\;T\neq 0}_{\chi\bar\chi\to ij}|^2 + \int d\Pi_\gamma \Big[f_\gamma \left(|\mathcal{M}_{\chi\bar\chi\to ij\gamma}|^2 + |\mathcal{M}_{\chi\bar\chi\gamma\to ij}|^2 \right)$$

$$- f_i \left(|\mathcal{M}_{\chi\bar\chi\to ij\gamma}|^2 + |\mathcal{M}_{\chi\bar\chi i\to j\gamma}|^2 \right) - f_j \left(|\mathcal{M}_{\chi\bar\chi\to ij\gamma}|^2 + |\mathcal{M}_{\chi\bar\chi j\to i\gamma}|^2 \right) \Big] \Big\}$$

thermal 1-loop

photon absorption

photon emission

SM fermions emission

SM fermions absorption

$$-f_i f_j \Big\{ |\mathcal{M}^{\rm LO}_{ij\to\chi\bar\chi}|^2 + |\mathcal{M}^{{\rm NLO}\;T=0}_{ij\to\chi\bar\chi}|^2 + \int d\Pi_\gamma |\mathcal{M}_{ij\to\chi\bar\chi\gamma}|^2 +$$

$$|\mathcal{M}^{{\rm NLO}\;T\neq 0}_{ij\to\chi\bar\chi}|^2 + \int d\Pi_\gamma \Big[f_\gamma \left(|\mathcal{M}_{ij\to\chi\bar\chi\gamma}|^2 + |\mathcal{M}_{ij\gamma\to\chi\bar\chi}|^2 \right)$$

$$- f_\chi \left(|\mathcal{M}_{ij\to\chi\bar\chi\gamma}|^2 + |\mathcal{M}_{ij\chi\to\chi\gamma}|^2 \right) - f_{\bar\chi} \left(|\mathcal{M}_{ij\to\chi\bar\chi\gamma}|^2 + |\mathcal{M}_{ij\bar\chi\to\bar\chi\gamma}|^2 \right) \Big] \Big\}.$$

QUESTIONS:

1. how the (soft and collinear) IR divergence cancellation happen?

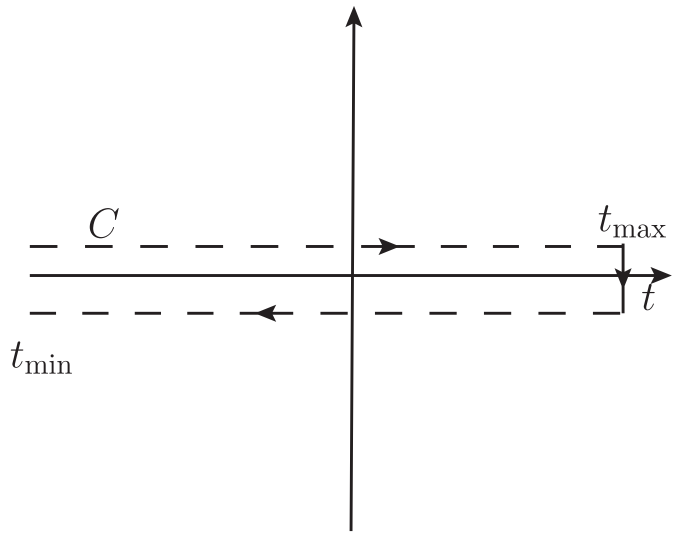
2. does Boltzmann equation itself receive quantum corrections?

3. how large are the remaining finite T corrections?

Program: develop a method for relic density calculation directly from QFT and free from IR problems

framework exists: non-equilibrium thermal field theory

CLOSED TIME PATH
FORMALISM



$$i\Delta(x,y) = \langle T_C \phi(x)\phi^\dagger(y)\rangle,$$
$$iS_{\alpha\beta}(x,y) = \langle T_C \psi_\alpha(x)\bar{\psi}_\beta(y)\rangle,$$

contour Green's functions obey Dyson-Schwinger eqs:

$$\Delta(x,y) = \Delta_0(x,y) - \int_C d^4z \int_C d^4z' \Delta_0(x,z)\Pi(z,z')\Delta(z',y),$$
$$S_{\alpha\beta}(x,y) = S^0_{\alpha\beta}(x,y) - \int_C d^4z \int_C d^4z' S^0_{\alpha\gamma}(x,z)\Sigma_{\gamma\rho}(z,z')S_{\rho\beta}(z',y),$$

which can be rewritten in the form of Kadanoff-Baym eqs:

$$(-\partial^2 - m_\phi^2)\Delta^{\lessgtr}(x,y) - \int d^4z \left(\Pi_h(x,z)\Delta^{\lessgtr}(z,y) - \Pi^{\lessgtr}(x,z)\Delta_h(z,y)\right) = \mathcal{C}_\phi,$$
$$(i\partial\!\!\!/ - m_\chi)S^{\lessgtr}(x,y) - \int d^4z \left(\Sigma_h(x,z)S^{\lessgtr}(z,y) - \Sigma^{\lessgtr}(x,z)S_h(z,y)\right) = \mathcal{C}_\chi,$$

Closed Time Path

path to Boltzmann equation

Kadanoff-Baym $\Longrightarrow$ Boltzmann

$$E\left(\partial_t - H\vec{p}\cdot\nabla_{\vec{p}}\right) f = \mathcal{C}[f].$$

collision term **derived** from thermal QFT

Assumptions:

weak inhomogeneity

gradient expansion

quasi-particle approx.

weak interactions

Justification:

inhomogeneity

plasma excitation momenta

$$\partial \ll k$$

freeze-out happens close to equilibrium

Closed Time Path

FORMALISM: COLLISION TERM

the fermion collision terms is defined as:

propagators

$$\mathcal{C}_\chi = \frac{1}{2}\int d^4 z \left(\Sigma^>(x,z) S^<(z,y) - \Sigma^<(x,z) S^>(z,y)\right)$$

self-energies

where the propagators:

thermal part

$$iS^c(p) = \frac{i\left(\not{p}+m\right)}{p^2-m^2+i\eta} - \boxed{2\pi\left(\not{p}+m\right)\delta\left(p^2-m^2\right) f\left(p^0\right)}$$

$$iS^a(p) = -\frac{i\left(\not{p}+m\right)}{p^2-m^2+i\eta} + 2\pi\left(\not{p}+m\right)\delta\left(p^2-m^2\right)\left(1-f\left(p^0\right)\right)$$

$$\left.\begin{aligned} iS^>(p) &= 2\pi\left(\not{p}+m\right)\delta\left(p^2-m^2\right)\left(1-f\left(p^0\right)\right) \\ iS^<(p) &= -2\pi\left(\not{p}+m\right)\delta\left(p^2-m^2\right) f\left(p^0\right) \end{aligned}\right\} \text{"cut" propagators}$$

the presence of distribution functions inside propagators $\Rightarrow$ known collision term structure

RESULTS

coming back to our example...

every contribution can be written in a form:

$$\int_0^\infty d\omega\, f_\gamma(\omega) \boxed{S(\omega, e_\chi, \epsilon, \xi)} \qquad f_\gamma(\omega) = \frac{1}{1 - e^{\omega/T}}$$

photon energy

expand in ω

$$S = \sum_{i=-1}^{\infty} s_n \omega^n$$

note:

$$J_n \equiv \int_0^\infty f_B(\omega)\omega^n d\omega = \begin{cases} \text{div} & n \leq 0 \\ \sim \tau^{n+1} & n > 0 \end{cases}$$

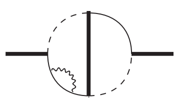
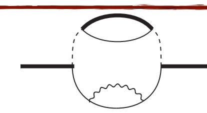
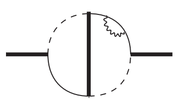
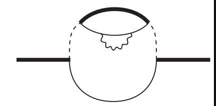
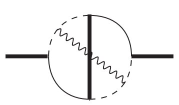
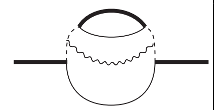
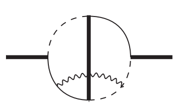
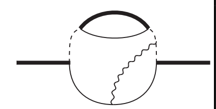
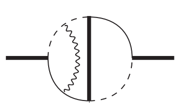
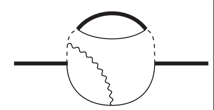
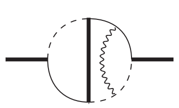
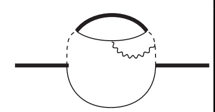
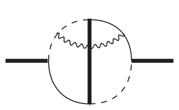
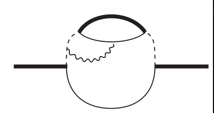
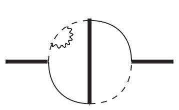
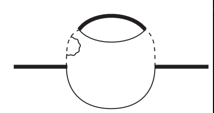
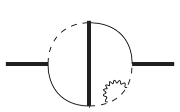
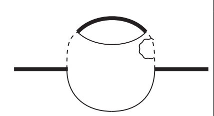
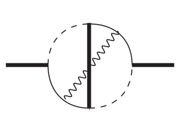
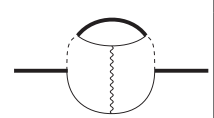
IR divergence in separate terms: $J_{-1} \leftrightarrow T = 0$ soft div

$J_0 \leftrightarrow T = 0$ soft eikonal

finite T corrections: $J_1 \leftrightarrow \mathcal{O}(\tau^2) \quad \ldots$

RESULTS

IR DIVERGENCE CANCELLATION: S-WAVE

The divergent part J_{-1}							
Type A	Real	Virtual	External	Type B	Real	Virtual	External
	$\frac{\alpha(1-2\epsilon^2)}{\pi\epsilon^2}$		$-\frac{\alpha(1-2\epsilon^2)}{\pi\epsilon^2}$		$-\frac{\alpha}{\pi\epsilon^2}$		$\frac{\alpha}{\pi\epsilon^2}$
	$\frac{\alpha(1-2\epsilon^2)}{\pi\epsilon^2}$		$-\frac{\alpha(1-2\epsilon^2)}{\pi\epsilon^2}$		$-\frac{\alpha}{\pi\epsilon^2}$		$\frac{\alpha}{\pi\epsilon^2}$
	0				0		
	0	0			0	0	
	0	0			0	0	
	0	0			0	0	
	0	0			0	0	
		0				0	
		0				0	
	$\frac{2\alpha(1-2\epsilon^2)^2}{\pi\epsilon^2\sqrt{1-4\epsilon^2}}L$	$-\frac{2\alpha(1-2\epsilon^2)^2}{\pi\epsilon^2\sqrt{1-4\epsilon^2}}L$			$\frac{2\alpha(1-2\epsilon^2)}{\pi\epsilon^2\sqrt{1-4\epsilon^2}}L$	$-\frac{2\alpha(1-2\epsilon^2)}{\pi\epsilon^2\sqrt{1-4\epsilon^2}}L$	

cancels in **every** row separately

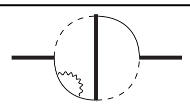
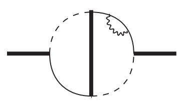
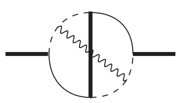
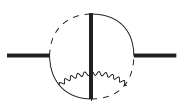
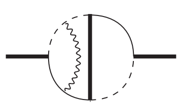
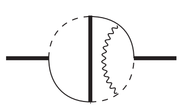
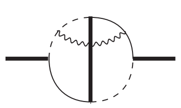
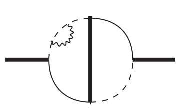
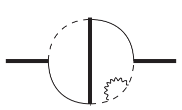
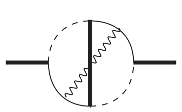
$\Longrightarrow$ every CTP self-energy is IR finite

Results

Finite T correction: s-wave

factorized $\frac{\pi}{6}\alpha\tau^2\frac{a_{\rm tree}}{\epsilon^2}$

The finite part J_1

Type A	Real	Virtual	External
	$\frac{2(1-\xi^2)}{D^2D_\xi^2}+\frac{(1-2\epsilon^2)p_1(\epsilon,\xi)}{2D^2D_\xi^2}\boxed{+\frac{1}{2\sqrt{D}}L}$		$\frac{(1-2\epsilon^2)(\xi^2-3D)}{2DD_\xi}\boxed{-\frac{1}{2\sqrt{D}}L}$
	— " —		— " —
	$-\frac{4(1-2\epsilon^2)D}{D_\xi^2}$		
	$-\frac{2(1-2\epsilon^2)\xi^2}{D_\xi^2}-\frac{f_1(\epsilon,\xi)}{\sqrt{D}D_\xi^2}L$	$\frac{2(1-2\epsilon^2)(D-\xi^2)}{D_\xi^2}+\frac{f_1(\epsilon,\xi)}{\sqrt{D}D_\xi^2}L$	
	— " —	— " —	
	— " —	— " —	
	— " —	— " —	
		$-\frac{4(1-2\epsilon^2)D}{D_\xi^2}$	
		— " —	
	$\frac{2(1-2\epsilon^2)p_2(\epsilon,\xi)+(1-\xi^2)^2}{D^2D_\xi^2}+\frac{4f_2(\epsilon,\xi)}{\sqrt{D}D_\xi^2}L$	$\frac{16\epsilon^2(2-3\epsilon^2)-(3-\xi^2)^2}{D_\xi^2}-\frac{4f_2(\epsilon,\xi)}{\sqrt{D}D_\xi^2}L$	

$\longrightarrow$ **Log** terms cancels in **every** row separately

$\Downarrow$

no collinear divergence!

$$\xi=\frac{m_\phi}{m_\chi}\gtrsim 1$$

$$\tau=\frac{T}{m_\chi}\ll 1$$

$$\epsilon=\frac{m_f}{2m_\chi}\ll\tau$$

separate contributions complicated, but when summed up...

$$\Delta a_{\tau^4}^{\epsilon=0}=\frac{8\pi^2\lambda^4\alpha\tau^4}{45}\frac{1}{(1+\xi^2)^4}=\frac{4\pi}{45}\alpha\tau^4\frac{1}{(1+\xi^2)^2}\frac{a_{\rm tree}}{\epsilon^2}\bigg|_{\epsilon=0}$$

strongly suppressed as at kinetic equilibrium $\tau\sim v^2$

THE POWER OF THERMAL OPE

M. Beneke, F. Dighera, AH, 1607.03910

The cross section can be written as the Im part of the forward scattering amplitude:

$$\sigma v_{\rm rel} = \frac{2}{s}\,{\rm Im}\left\{(-i)\boxed{\int d^4x\, \frac{1}{4}\sum_{\rm spin}\langle\bar\chi\chi;T|\,\mathcal{T}\left\{\mathcal{O}_{\rm ann}(0)\,\mathcal{O}^\dagger_{\rm ann}(x)\right\}\,|\bar\chi\chi;T\rangle}\right\}$$

clear separation of soft (thermal effects) and hard (annihilation/decay) modes

$$T \ll m$$

$\Rightarrow$ **Operator Product Expansion**

$$-i\int d^4x\; e^{-ip\cdot x}\;\mathcal{T}\left\{J^\mu_A(0)\,J^{\nu\dagger}_B(x)\right\} = \sum_i C^i_{AB}(p)\cdot\mathcal{O}_i$$

Wilson coeffs. matched at T=0

Possible operators up to dim 4:

$$\mathbb{1}\,, \qquad F^{\alpha\beta}F^{\gamma\delta}\,, \qquad m_f\bar f\,\Gamma f\,, \qquad \bar f\,\Gamma\, iD^\alpha f$$

Matrix elements: LO $\quad\mathcal{O}(\alpha T^4)\quad \mathcal{O}(\alpha m_f^2 T^2)\quad \mathcal{O}(\alpha T^4)$

No dim 2 operator! No IR divergence to begin with!

Advantages of OPE

- The scaling with T is manifest
- Separation of T=0 and T-dependent contributions
- Significant simplification of the computations
- Clear physics interpretation: at $\mathcal{O}(\alpha\tau^2)$ effects of thermal kinetic energy

Example: muon decay in thermal bath*

Czarnecki et al. '11

$$-i\int d^4x\, e^{-ip\cdot x}\,\mathcal{T}\{J^\mu(0)\,J^{\nu\dagger}(x)\} = C_0^{\mu\nu}\bar{\psi}\psi + \cancel{C_2^{\mu\nu}\bar{\psi}\frac{i}{2}\sigma_{\alpha\beta}F^{\alpha\beta}\psi} + \mathcal{O}(m_\psi^{-3}),$$

$$\bar{\psi}\psi = \bar{\psi}\not{v}\psi + \boxed{\frac{1}{2m_\psi^2}\bar{\psi}\,(iD_\perp)^2\,\psi} + \cancel{\frac{i}{4m_\psi^2}\bar{\psi}\sigma_{\alpha\beta}F^{\alpha\beta}\psi} + \mathcal{O}(m_\psi^{-3}),$$

LO $\quad\mathcal{O}(\alpha\tau^2)$

...and the final correction:

$$\Gamma_T = \Gamma_0\,(1 - \boxed{K_\psi}) + \mathcal{O}(T^3/m_\psi^3).$$

*Analogy: semi-leptonic H_b decay in QCD

In the Literature:

OPE in finite temperature - Hatsuda, Koike, Lee '93; Mallik '97; ...

Related EFT approach - Biondini, Brambilla, Escobedo, Vairo '13; ...

SUMMARY: PART II

1. how the (soft and collinear) IR divergence cancellation happen?
   automatic in thermal QFT formalism, cancellation at the level of every CTP self-energy

2. does Boltzmann equation itself receive quantum corrections?
   no, not at NLO

3. how large are the remaining finite T corrections?
   strongly suppressed, of order $\mathcal{O}(\alpha T^4)$

Exception *N+1*:
LO sometimes is not enough
(and then in principle $T \neq 0$ QFT needed)

...but in practice one can safely use BE with NLO cross-section

BACKUP

Relic Density

Three Exceptions

Griest & Seckel PRD'91

1. Co-annihilations

if more than one state share a conserved quantum number making DM stable

$$\langle \sigma_{\rm eff} \mathrm{v} \rangle = \sum_{ij} \langle \sigma_{ij} \mathrm{v}_{ij} \rangle \frac{n_i^{\rm eq} n_j^{\rm eq}}{n_{\rm eq}^2}$$

with: $\sigma_{ij} = \sum_X \sigma(\chi_i \chi_j \to X)$

e.g., SUSY

2. Annihilation to forbidden channels

if DM is slightly below mass threshold for annihilation $\Longrightarrow$ „forbidden" channel can still be accessible in thermal bath

recent e.g., 1505.07107

3. Annihilation near poles

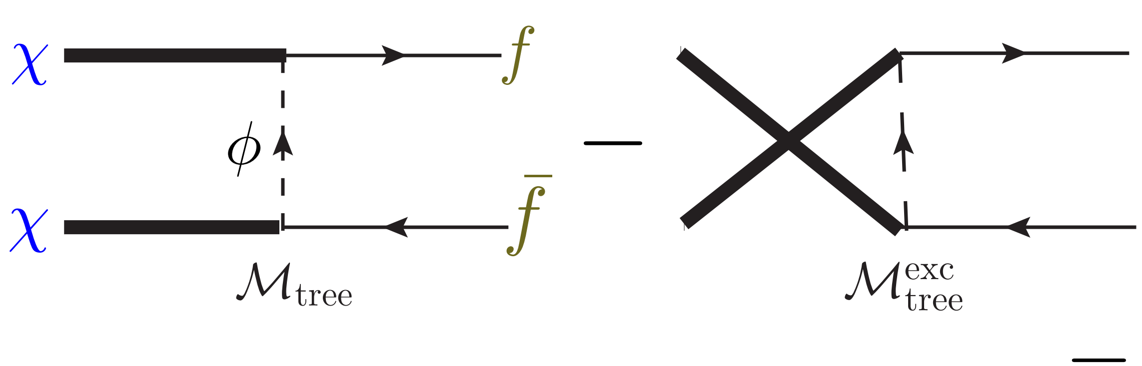
expansion in velocity (s-wave, p-wave, etc.) not safe

(more historical issue: these days most people use numerical codes)

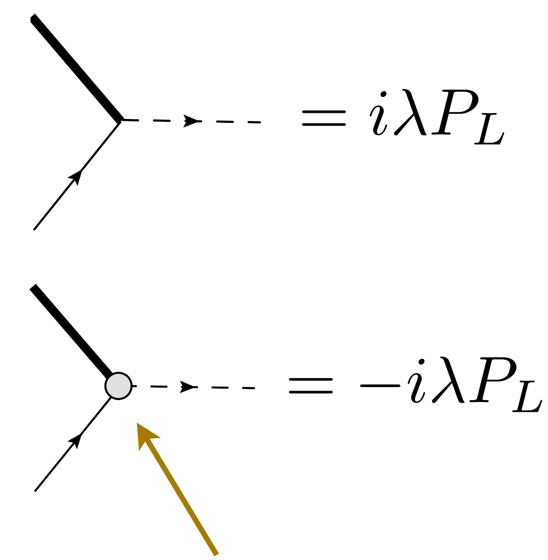
COLLISION TERM
EXAMPLE

Bino-like DM: χ Majorana fermion, SM singlet

annihilation process at tree level:



vertices (2 types):



dotted

scale hierarchy: $\boxed{m_\phi \gtrsim m_\chi} \gg T \gg \boxed{m_f}$

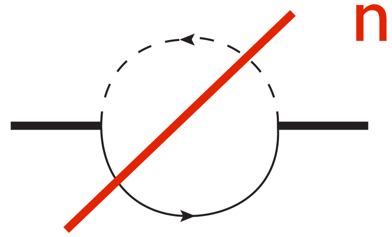
no thermal contributions

effectively massless

rescaled variables: $\tau = \dfrac{T}{m_\chi} \ll 1 \qquad \epsilon = \dfrac{m_f}{2m_\chi} \ll \tau \qquad \xi = \dfrac{m_\phi}{m_\chi} \gtrsim 1$

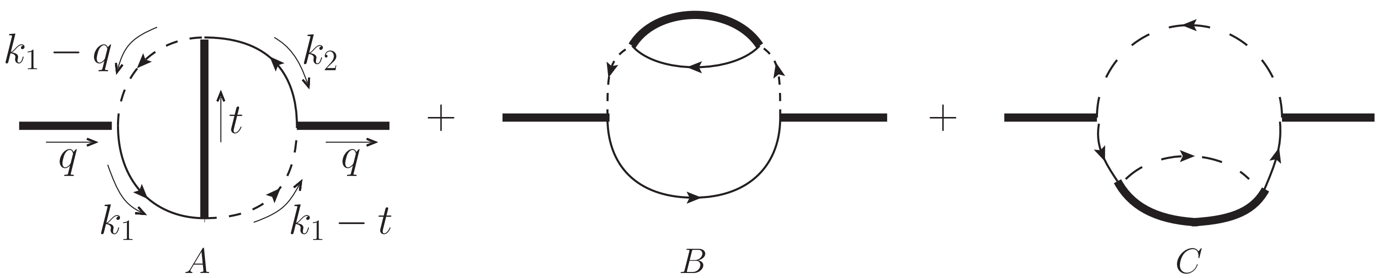
Collision Term

computation

$i\Sigma_1 =$ 

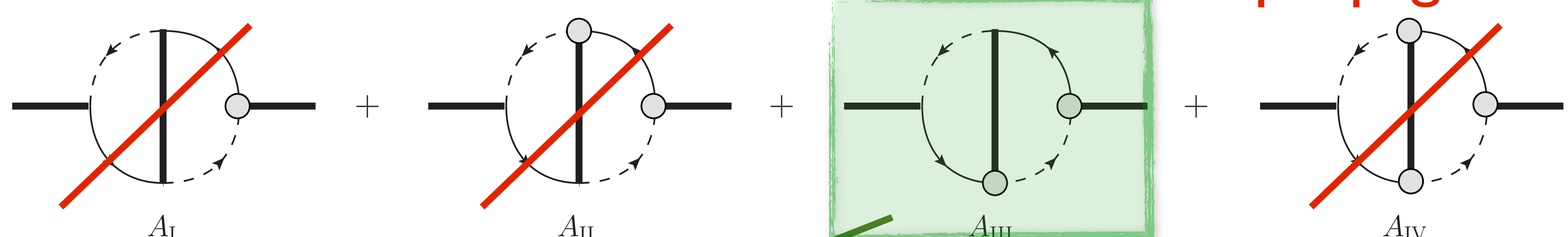
no # changing processes

summed over dotted and undotted indices

$i\Sigma_2 =$ 

$\Downarrow$

cut scalar propagator

$i\Sigma_A^> =$ 

$$\Sigma^>_{A_{\rm III}}(q)\,S^<(q) = -\lambda^4 \int \frac{d^4t}{(2\pi)^4}\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}(2\pi)^4\,\delta\left(q+t-k_1-k_2\right)$$
$$\underbrace{i\Delta^{11}(k_1-q)\,i\Delta^{22}(k_1-t)}_{\equiv\mathcal{S}}\underbrace{P_{\rm R} iS^{21}(k_2)\,P_{\rm L} iS^{12}(t)\,P_{\rm L} iS^{21}(k_1)\,P_{\rm R} iS^{12}(q)}_{\equiv\mathcal{F}}$$

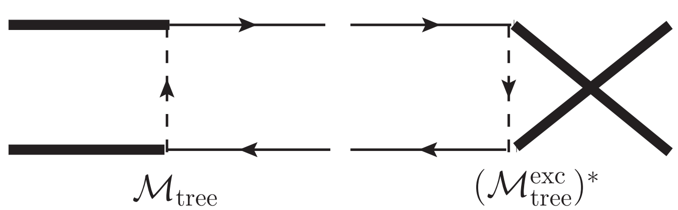
Collision Term
matching

after inserting the propagators:

$$\Sigma^{>}_{A_{\rm III}}\left(q\right) S^{<}\left(q\right) = \frac{1}{2E_{\chi_1}}\left(2\pi\right)\delta\left(q^0 - E_{\chi_1}\right)\int \frac{d^4 t}{\left(2\pi\right)^3 2E_{\chi_2}}\delta\left(t^0 - E_{\chi_2}\right)\times$$

$$\int \frac{d^3\vec{k}_1}{\left(2\pi\right)^3 2E_{f_1}}\frac{d^3\vec{k}_2}{\left(2\pi\right)^3 2E_{f_2}}\left(2\pi\right)^4\delta\left(q+t-k_1-k_2\right)\left|\mathcal{M}_A\right|^2\left[f_\chi\left(q\right) f_\chi\left(t\right)\left(1-f_f^{\rm eq}\left(k_1^0\right)\right)\left(1-f_f^{\rm eq}\left(k_2^0\right)\right)\right]$$

$\Rightarrow$ one indeed recovers the known collision term and

$$\left|\mathcal{M}_A\right|^2 =$$ 

(part of) tree level $\left|\mathcal{M}\right|^2$

repeating the same for B type diagrams the bottom line:

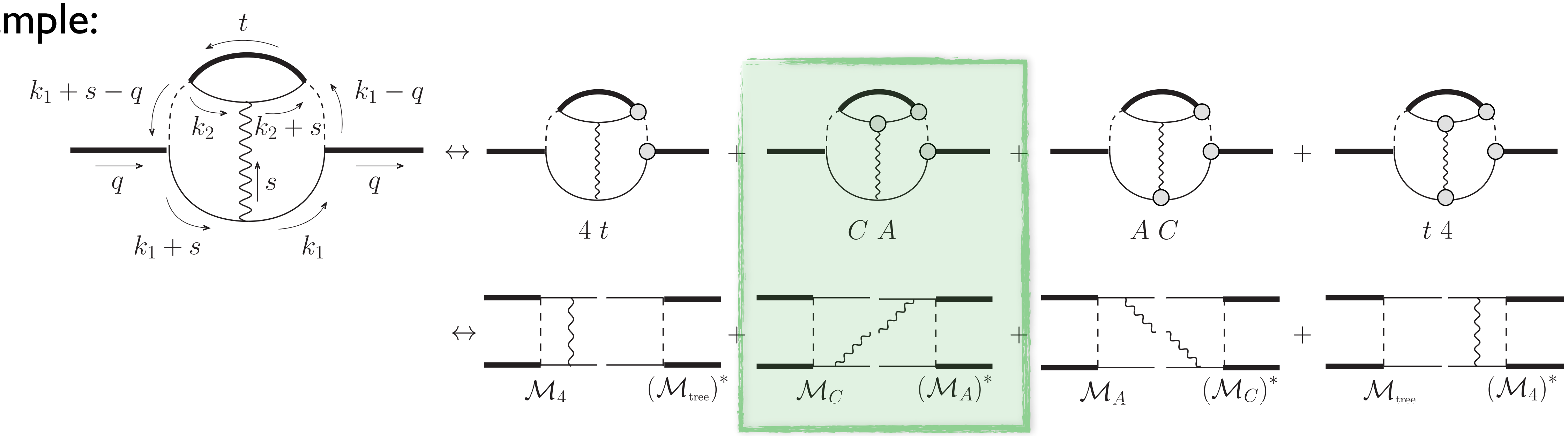
$i\Sigma^{>} \leftrightarrow$ tree level annihilation contribution to the collision term

Collision Term

matching at NLO

$i\Sigma_3 =$ 20 self-energy diagrams

example:



$$\Sigma^{>}_{\text{CA}}(q)\, S^{<}(q) = \frac{1}{2E_{\chi_1}}(2\pi)\,\delta\left(q^0 - E_{\chi_1}\right)\int \frac{d^4 t}{(2\pi)^3\, 2E_{\chi_2}}\delta\left(t^0 - E_{\chi_2}\right)$$

$$\int \frac{d^3\vec{k}_1}{(2\pi)^3\, 2E_{f_1}}\frac{d^3\vec{k}_2}{(2\pi)^3\, 2E_{f_2}}\frac{d^3\vec{s}}{(2\pi)^3\, 2E_{\gamma}}(2\pi)^4\,\delta\left(q + t - k_1 - k_2 - s\right)$$

$$\mathcal{M}_C\left(\mathcal{M}_A\right)^*\left[f_\chi(q)\, f_\chi(t)\left(1 - f_f^{\text{eq}}\left(k_1^0\right)\right)\left(1 - f_f^{\text{eq}}\left(k_2^0\right)\right)\left(1 + f_\gamma^{\text{eq}}\left(s^0\right)\right)\right]$$

$\Rightarrow$ at NLO thermal effects do **not** change the collision therm structure

What are the relevant Rates?

Around freeze-out, typically:

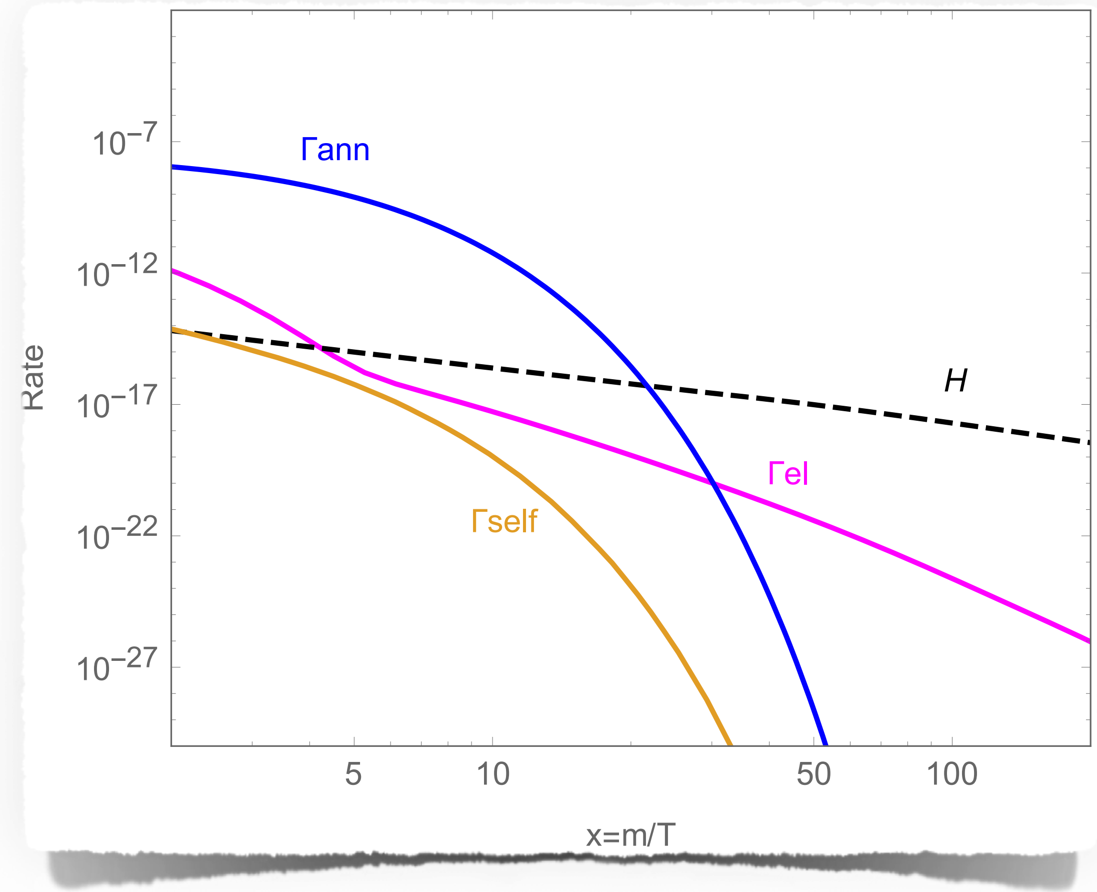
scattering Hubble annihilation

$$\Gamma_{\rm el} \gg H \sim \Gamma_{\rm ann}$$

what if

$$H \sim \Gamma_{\rm ann} \gtrsim \Gamma_{\rm el}$$

instead?



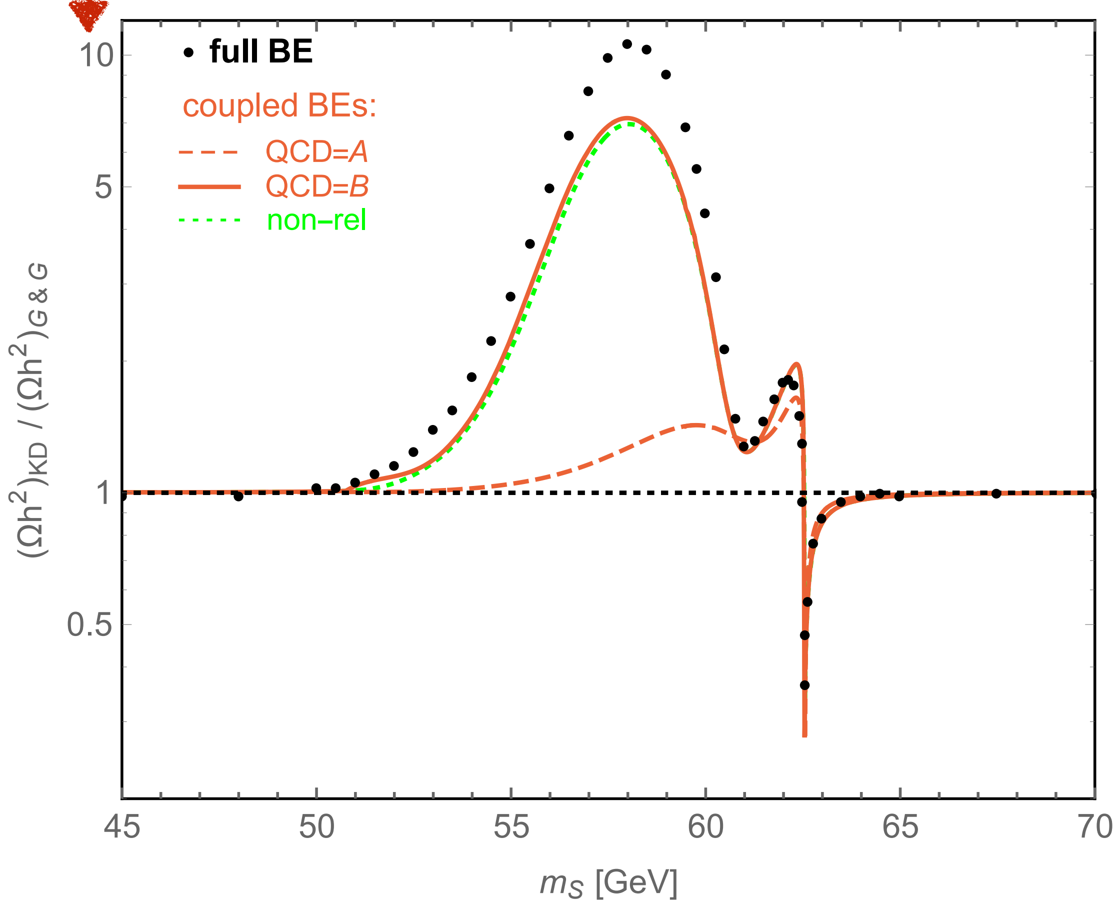
- can the kinetic equilibrium be still maintained?
- what can be the size of departure from $f_\chi^{\rm eq}$?
- how does this impact $\langle \sigma_{\chi\bar\chi \to ij} v_{\rm rel} \rangle$ and the DM density evolution?

Results

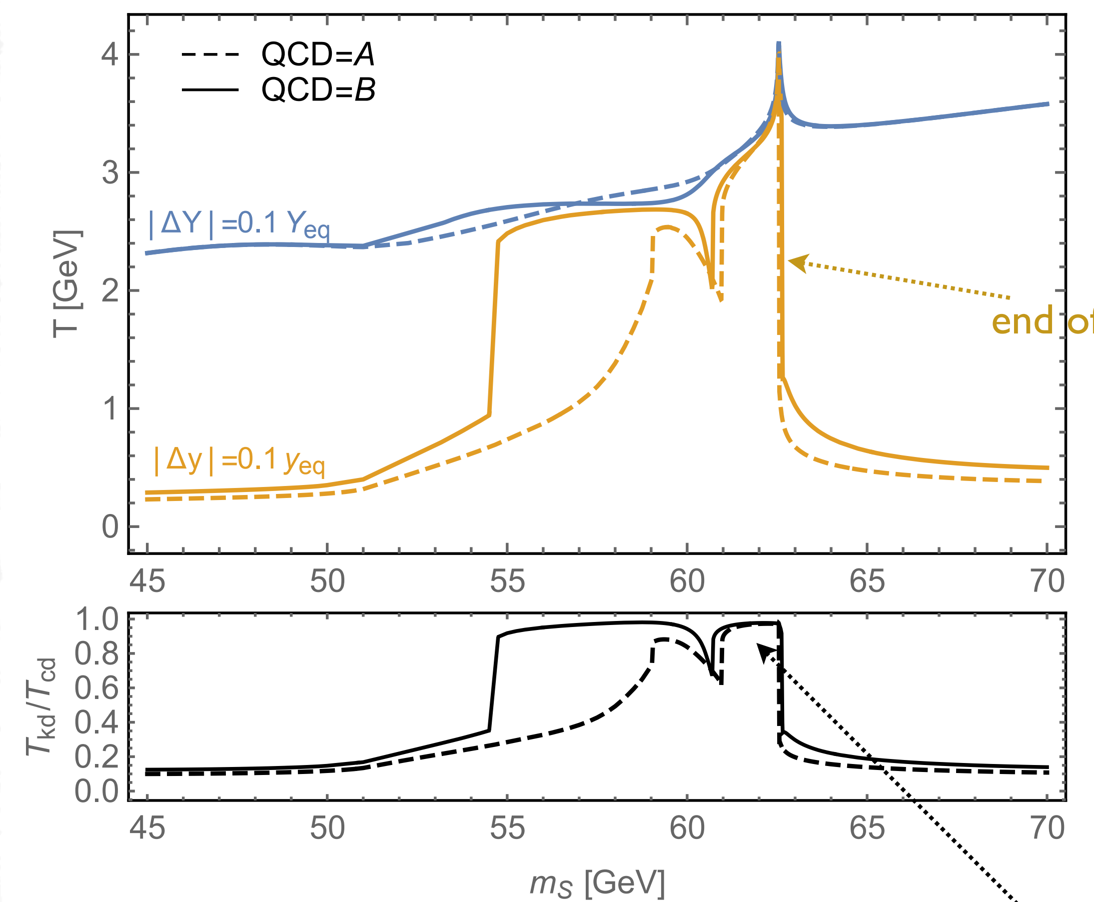
Effect

effect on relic density:
up to O(~10)

effect on relic density:



kinetic and chemical decoupling:



approaches 1, but does not reach it!

Why such non-trivial shape of the effect of early kinetic decoupling?

→ Let's inspect the *y* and *Y* evolution…